One-and-a-halfth-order Logic

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Motivation

Consider the following valid assertions in first-order logic:

- $\phi \supset \psi \supset \phi$
- if $a \notin fn(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin fn(\phi)$ then $\phi \supset \phi[\![a \mapsto t]\!]$
- if $b \notin fn(\phi)$ then $\forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]$

These are *not valid syntax* in first-order logic, because of *meta-level concepts*:

- meta-variables *varying* over syntax: ϕ, ψ, a, b, t
- properties of syntax: $a \notin fn(\phi)$, $\phi[\![a \mapsto t]\!]$, α-equivalence

Is there a logic in which the above assertions can be expressed directly in the syntax?

Motivation (2)

Consider the following derivations in Gentzen's sequent calculus:

$$\dfrac{\dfrac{\dfrac{}{\psi, \phi \vdash \phi}\,(\mathbf{Ax})}{\phi \vdash \psi \supset \phi}\,(\supset\mathbf{R})}{\vdash \phi \supset \psi \supset \phi}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{p}(d), \mathsf{p}(c) \vdash \mathsf{p}(c)}\,(\mathbf{Ax})}{\mathsf{p}(c) \vdash \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\mathbf{R})}{\vdash \mathsf{p}(c) \supset \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\mathbf{R})$$

And for $b \notin fn(\phi)$:

$$\dfrac{\dfrac{}{\forall a.\phi \vdash \forall b.\phi[\![a \mapsto b]\!]}\,(\mathbf{Ax})}{\vdash \forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{}{\forall c.\mathsf{p}(c) \vdash \forall d.\mathsf{p}(d)}\,(\mathbf{Ax})}{\vdash \forall c.\mathsf{p}(c) \supset \forall d.\mathsf{p}(d)}\,(\supset\mathbf{R})$$

The left ones are not derivations, they are *schemas* of derivations.
When p is a *specific* atomic predicate and c and d are *specific* variables, the right ones are derivations; they are *instances* of the schemas on the left.

Is there a logic in which the derivation on the left is a derivation too?

Motivation (3)

First-order logic and its proof systems formalise *reasoning*.

But also a lot of reasoning is *about* first-order logic.

So why shouldn't that be formalised?

One-and-a-halfth-order logic does this by means of:

- formalising meta-variables;
- making properties of syntax explicit.

Overview

- Introduction to one-and-a-halfth-order logic
- Syntax of one-and-a-halfth-order logic
- Sequent calculus for one-and-a-halfth-order logic
- Axiomatisation of one-and-a-halfth-order logic
- Relation to first-order logic
- Semantics of one-and-a-halfth-order logic
- Conclusions, related and future work

Introduction

In the syntax of one-and-a-halfth-order logic:

- *Unknowns* P, Q and T represent meta-variables ϕ, ψ and t.
- *Atoms* a and b represent meta-variables a and b.
- *Freshness* $a\#P$ represents $a \notin fn(\phi)$.
- *Explicit substitution* $P[a \mapsto T]$ represents $\phi[\![a \mapsto t]\!]$.

Introduction (2)

The meta-level assertions in first-order logic

- $\phi \supset \psi \supset \phi$
- if $a \notin fn(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin fn(\phi)$ then $\phi \supset \phi[\![a \mapsto t]\!]$
- if $b \notin fn(\phi)$ then $\forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]$

correspond to valid assertions in the syntax of one-and-a-halfth-order logic:

- $P \supset Q \supset P$
- $a\#P \rightarrow P \supset \forall[a]P$
- $a\#P \rightarrow P \supset P[a \mapsto T]$
- $b\#P \rightarrow \forall[a]P \supset \forall[b]P[a \mapsto b]$

Introduction (3)

In sequent derivations of one-and-a-halfth-order logic:

- *Contexts of freshnesses* are added to the sequents.
- *Derivability of freshnesses* are added as side-conditions.
- *Substitutional equivalence on terms* is added as two derivation rules, taking care of α-equivalence and substitution.

Introduction (4)

The (schematic) derivations in first-order logic

$$\dfrac{\dfrac{\dfrac{}{\psi, \phi \vdash \phi}\,(\mathbf{Ax})}{\phi \vdash \psi \supset \phi}\,(\supset\mathbf{R})}{\vdash \phi \supset \psi \supset \phi}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{p}(d), \mathsf{p}(c) \vdash \mathsf{p}(c)}\,(\mathbf{Ax})}{\mathsf{p}(c) \vdash \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\mathbf{R})}{\vdash \mathsf{p}(c) \supset \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\mathbf{R})$$

correspond to valid derivations in one-and-a-halfth-order logic:

$$\dfrac{\dfrac{\dfrac{}{Q, P \vdash_{\emptyset} P}\,(\mathbf{Ax})}{P \vdash_{\emptyset} Q \supset P}\,(\supset\mathbf{R})}{\vdash_{\emptyset} P \supset Q \supset P}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{p}(d), \mathsf{p}(c) \vdash_{\emptyset} \mathsf{p}(c)}\,(\mathbf{Ax})}{\mathsf{p}(c) \vdash_{\emptyset} \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\mathbf{R})}{\vdash_{\emptyset} \mathsf{p}(c) \supset \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\mathbf{R})$$

Introduction (5)

The (schematic) derivations in first-order logic, where $b \notin fn(\phi)$,

$$\dfrac{\dfrac{}{\forall a.\phi \vdash \forall b.\phi[\![a \mapsto b]\!]}\,(\mathbf{Ax})}{\vdash \forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{}{\forall c.\mathsf{p}(c) \vdash \forall d.\mathsf{p}(d)}\,(\mathbf{Ax})}{\vdash \forall c.\mathsf{p}(c) \supset \forall d.\mathsf{p}(d)}\,(\supset\mathbf{R})$$

correspond to valid derivations in one-and-a-halfth-order logic:

$$\dfrac{\dfrac{\dfrac{}{\forall[a]P \vdash_{b\#P} \forall[a]P}\,(\mathbf{Ax})}{\forall[a]P \vdash_{b\#P} \forall[b]P[a \mapsto b]}\,(\mathbf{StructR})}{\vdash_{b\#P} \forall[a]P \supset \forall[b]P[a \mapsto b]}\,(\supset\mathbf{R}) \qquad (b\#P \vdash_{\mathsf{SUB}} \forall[a]P = \forall[b]P[a \mapsto b])$$

$$\dfrac{\dfrac{\dfrac{}{\forall[c]\mathsf{p}(c) \vdash_{\emptyset} \forall[c]\mathsf{p}(c)}\,(\mathbf{Ax})}{\forall[c]\mathsf{p}(c) \vdash_{\emptyset} \forall[d]\mathsf{p}(d)}\,(\mathbf{StructR})}{\vdash_{\emptyset} \forall[c]\mathsf{p}(c) \supset \forall[d]\mathsf{p}(d)}\,(\supset\mathbf{R}) \qquad (\emptyset \vdash_{\mathsf{SUB}} \forall[c]\mathsf{p}(c) = \forall[d]\mathsf{p}(d))$$

Syntax of one-and-a-halfth-order logic

We use **Nominal Terms** to specify the syntax, since they have built-in support for: *meta-variables*, *freshness* and *binding*.

Nominal terms allow for a *direct* and *natural* representation of systems with binding.

Nominal terms are *first-order*, not higher-order.

Timeline of nominal terms:

- FM Set Theory (Gabbay, Pitts)
- Nominal Sets (Gabbay, Pitts)
- Nominal Terms (Urban, Pitts, Gabbay)
- Nominal Rewriting (Fernández, Gabbay)
- Nominal Algebra (Gabbay, Mathijssen)

Sorts

Base sorts $\mathbb{P}$ for 'predicates' and $\mathbb{T}$ for 'terms'.

Atomic sort $\mathbb{A}$ for the object-level variables.

Sorts τ:

$$\tau ::= \mathbb{P} \mid \mathbb{T} \mid \mathbb{A} \mid [\mathbb{A}]\tau$$

Terms

Atoms $a, b, c, \ldots$ have sort $\mathbb{A}$; they represent *object-level* variable symbols.

Unknowns $X, Y, Z, \ldots$ have sort τ; they represent *meta-level* variable symbols.
Let P, Q, R be unknowns of sort $\mathbb{P}$, and T, U of sort $\mathbb{T}$.

We call $\pi \cdot X$ a **moderated unknown**.
This represents the **permutation of atoms** π acting on an unknown term.

Term-formers f_ρ have an associated **arity** $\rho = (\tau_1, \ldots, \tau_n)\tau$.
$\mathsf{f} : \rho$ means ‘f with arity ρ’.

Terms t, subscripts indicate sorting rules:

$$t \ ::= a_{\mathbb{A}} \mid (\pi \cdot X_\tau)_\tau \mid ([a_{\mathbb{A}}]t_\tau)_{[\mathbb{A}]\tau} \mid (\mathsf{f}_{(\tau_1,\ldots,\tau_n)\tau}(t^1_{\tau_1}, \ldots, t^n_{\tau_n}))_\tau$$

Write f for $\mathsf{f}()$ if $n = 0$.

Terms (2)

Term-formers for one-and-a-halfth-order logic:

- $\bot : ()\mathbb{P}$ represents *falsity*;
- $\supset : (\mathbb{P}, \mathbb{P})\mathbb{P}$ represents *implication*, write $\phi \supset \psi$ for $\supset(\phi, \psi)$;
- $\forall : ([\mathbb{A}]\mathbb{P})\mathbb{P}$ represents *universal quantification*, write $\forall[a]\phi$ for $\forall([a]\phi)$;
- $\approx : (\mathbb{T}, \mathbb{T})\mathbb{P}$ represents *object-level equality*, write $t \approx u$ for $\approx(t, u)$;
- $\mathsf{var} : (\mathbb{A})\mathbb{T}$ is *variable casting*, forced upon us by the sort system, write a for $var(a)$;
- $\mathsf{sub} : ([\mathbb{A}]\tau, \mathbb{T})\tau$, where $\tau \in \{\mathbb{T}, [\mathbb{A}]\mathbb{T}, \mathbb{P}, [\mathbb{A}]\mathbb{P}\}$, is *explicit substitution*, write $v[a \mapsto t]$ for $\mathsf{sub}([a]v, t)$;
- $\mathsf{p}_1, \ldots, \mathsf{p}_n : (\mathbb{T}, \ldots, \mathbb{T})\mathbb{P}$ are *object-level predicate term-formers*;
- $\mathsf{f}_1, \ldots, \mathsf{f}_m : (\mathbb{T}, \ldots, \mathbb{T})\mathbb{T}$ are *object-level term-formers*.

Terms (3)

Sugar:

$$\top \text{ is } \bot \supset \bot \qquad \neg\phi \text{ is } \phi \supset \bot \qquad \phi \wedge \psi \text{ is } \neg(\phi \supset \neg\psi)$$
$$\phi \vee \psi \text{ is } \neg\phi \supset \psi \qquad \phi \Leftrightarrow \psi \text{ is } (\phi \supset \psi) \wedge (\psi \supset \phi) \qquad \exists[a]\phi \text{ is } \neg\forall[a]\phi$$

Descending order of operator precedence:

$$[a]\_,\ \_[\_ \mapsto \_],\ \approx,\ \{\neg, \forall, \exists\},\ \{\wedge, \vee\},\ \supset,\ \Leftrightarrow$$

$\wedge$, $\vee$, $\supset$ and $\Leftrightarrow$ associate to the right.

Example terms of sort $\mathbb{P}$:

$$P \supset Q \supset P \qquad P \supset \forall[a]P \qquad P \supset P[a \mapsto T] \qquad \forall[a]P \supset \forall[b]P[a \mapsto b]$$

Freshness

Freshness (assertions) $a\#t$, which means 'a is fresh for t.
If t is an unknown X, the freshness is called **primitive**.

A **freshness context** Δ is a set of *primitive* freshnesses.

Example freshness contexts:

$$\emptyset \qquad a\#X \qquad a\#P, b\#Q \qquad a\#P, Q$$

We call $\Delta \rightarrow t$ a **term-in-context**.
We may write t if $\Delta = \emptyset$.

Example terms-in-context of sort $\mathbb{P}$:

$$P \supset Q \supset P \qquad a\#P \rightarrow P \supset \forall[a]P$$
$$a\#P \rightarrow P \supset P[a \mapsto T] \qquad b\#P \rightarrow \forall[a]P \supset \forall[b]P[a \mapsto b]$$

Derivability of freshness

$$\frac{}{a\#b}\,(\#\mathbf{ab}) \qquad \frac{\pi^{-1}(a)\#X}{a\#\pi\cdot X}\,(\#\mathbf{X})$$

$$\frac{}{a\#[a]t}\,(\#[]\mathbf{a}) \qquad \frac{a\#t}{a\#[b]t}\,(\#[]\mathbf{b}) \qquad \frac{a\#t_1 \quad \cdots \quad a\#t_n}{a\#\mathsf{f}(t_1,\ldots,t_n)}\,(\#\mathbf{f})$$

a and b range over distinct atoms.

Write $\Delta \vdash a\#t$ when there exists a derivation of $a\#t$ using the elements of Δ as assumptions. Say that $a\#t$ **is derivable from** Δ.

Examples:

$$\vdash a\#\forall[a]P \qquad a\#P \vdash a\#\forall[b]P \qquad a\#T, U \vdash a\#\ T \approx U$$

Derivability of equality

Equality (assertions) $t = u$, where t and u are of the same sort.

Nominal Algebra is the logic of equality between nominal terms.

Derivability:

$$\frac{}{t = t}\,(\mathbf{refl}) \qquad \frac{t = u}{u = t}\,(\mathbf{symm}) \qquad \frac{t = u \quad u = v}{t = v}\,(\mathbf{tran})$$

$$\frac{t = u}{C[t] = C[u]}\,(\mathbf{cong}) \qquad \frac{a\#t \quad b\#t}{(a\ b)\cdot t = t}\,(\mathbf{perm})$$

$$\frac{\Delta^\pi \sigma}{t^\pi \sigma = u^\pi \sigma}\,(\mathbf{ax_A})\ A \text{ is } \Delta \to t = u \qquad \begin{array}{c} [a\#X_1, \ldots, a\#X_n] \qquad \Delta \\ \vdots \\ \dfrac{t = u}{t = u}\,(\mathbf{fr}) \quad (a \notin t, u, \Delta) \end{array}$$

Write $\Delta \vdash_{\mathsf{T}} t = u$ when $t = u$ **is derivable from** Δ using **axioms** A from T only.

Derivability of equality (2)

Nominal algebraic theory SUB of explicit substitution:

$$
\begin{array}{rr}
(\mathbf{var}\mapsto) & a[a\mapsto T]=T \\
(\#\mapsto) & a\#X\to X[a\mapsto T]=X \\
(\mathbf{f}\mapsto) & \mathsf{f}(X_1,\ldots,X_n)[a\mapsto T]=\mathsf{f}(X_1[a\mapsto T],\ldots,X_n[a\mapsto T]) \\
(\mathbf{abs}\mapsto) & b\#T\to([b]X)[a\mapsto T]=[b](X[a\mapsto T]) \\
(\mathbf{ren}\mapsto) & b\#X\to X[a\mapsto b]=(b\ a)\cdot X
\end{array}
$$

Examples:

$$
\begin{array}{c}
b\#P\vdash_{\mathsf{SUB}}\forall[a]P=\forall[b]P[a\mapsto b] \\
\vdash_{\mathsf{SUB}} X[a\mapsto a]=X \\
a\#Y\vdash_{\mathsf{SUB}} Z[a\mapsto X][b\mapsto Y]=Z[b\mapsto Y][a\mapsto X[b\mapsto Y]]
\end{array}
$$

Sequent calculus for one-and-a-halfth-order logic

We may call terms of sort $\mathbb{P}$ **predicates**, and denote them by ϕ and ψ.

Let **(predicate) contexts** Φ, Ψ be finite sets of predicates.
Examples:

$$\emptyset \qquad \phi \qquad \phi, \Phi \qquad \Phi, \Phi'$$

A **sequent** is a triple $\Phi \vdash_{\vartriangle} \Psi$.
We may omit empty predicate contexts, e.g. writing $\vdash_{\vartriangle}$ for $\emptyset \vdash_{\vartriangle} \emptyset$.

Define derivability on sequents...

Sequent calculus (2)

Rules resembling Gentzen's sequent calculus for first-order logic:

$$\frac{}{\phi,\ \Phi \vdash_{\vartriangle} \Psi,\ \phi}\ (\mathbf{Ax}) \qquad \frac{}{\bot,\ \Phi \vdash_{\vartriangle} \Psi}\ (\bot\mathbf{L})$$

$$\frac{\Phi \vdash_{\vartriangle} \Psi,\ \phi \quad \psi,\ \Phi \vdash_{\vartriangle} \Psi}{\phi \supset \psi,\ \Phi \vdash_{\vartriangle} \Psi}\ (\supset\mathbf{L}) \qquad \frac{\phi,\ \Phi \vdash_{\vartriangle} \Psi,\ \psi}{\Phi \vdash_{\vartriangle} \Psi,\ \phi \supset \psi}\ (\supset\mathbf{R})$$

$$\frac{\phi[a \mapsto t],\ \Phi \vdash_{\vartriangle} \Psi}{\forall[a]\phi,\ \Phi \vdash_{\vartriangle} \Psi}\ (\forall\mathbf{L}) \qquad \frac{\Phi \vdash_{\vartriangle} \Psi,\ \psi}{\Phi \vdash_{\vartriangle} \Psi,\ \forall[a]\psi}\ (\forall\mathbf{R}) \quad (\Delta \vdash a\#\Phi, \Psi)$$

$$\frac{\phi[a \mapsto t'],\ \Phi \vdash_{\vartriangle} \Psi}{t' \approx t,\ \phi[a \mapsto t],\ \Phi \vdash_{\vartriangle} \Psi}\ (\approx\mathbf{L}) \qquad \frac{}{\Phi \vdash_{\vartriangle} \Psi,\ t \approx t}\ (\approx\mathbf{R})$$

Sequent calculus (3)

Other rules:

$$\frac{\phi',\ \Phi \vdash_{\Delta} \Psi}{\phi,\ \Phi \vdash_{\Delta} \Psi}\ (\textbf{StructL}) \quad (\Delta \vdash_{\mathsf{SUB}} \phi' = \phi)$$

$$\frac{\Phi \vdash_{\Delta} \Psi,\ \psi'}{\Phi \vdash_{\Delta} \Psi,\ \psi}\ (\textbf{StructR}) \quad (\Delta \vdash_{\mathsf{SUB}} \psi' = \psi)$$

$$\frac{\Phi \vdash_{\Delta \cup \{a\#X_1,\ldots,a\#X_n\}} \Psi}{\Phi \vdash_{\Delta} \Psi}\ (\textbf{Fresh}) \quad (a \notin \Phi, \Psi, \Delta)$$

$$\frac{\Phi \vdash_{\Delta} \Psi,\ \phi \qquad \phi',\ \Phi \vdash_{\Delta} \Psi}{\Phi \vdash_{\Delta} \Psi}\ (\textbf{Cut}) \quad (\Delta \vdash_{\mathsf{SUB}} \phi = \phi')$$

Example derivations

Derivation of $a\#P \to P \supset \forall[a]P$:

$$\dfrac{\dfrac{\dfrac{}{P \vdash_{a\#P} P}\,(\mathbf{Ax})}{P \vdash_{a\#P} \forall\boxed{a}P}\,(\forall\mathbf{R}) \quad (a\#P \vdash a\#P)}{\vdash_{a\#P} P \supset \forall[a]P}\,(\supset\mathbf{R})$$

Derivation of $a\#P \to P \supset P[a \mapsto T]$:

$$\dfrac{\dfrac{\dfrac{}{P \vdash_{a\#P} P}\,(\mathbf{Ax})}{P \vdash_{a\#P} P\boxed{a \mapsto T}}\,(\mathbf{StructR}) \quad (a\#P \vdash_{\mathsf{SUB}} P = P[a \mapsto T])}{\vdash_{a\#P} P \supset P[a \mapsto T]}\,(\supset\mathbf{R})$$

Properties of the sequent calculus

We may *permute* atoms and *instantiate* unknowns in derivations.

Theorem 1 If Π is a valid derivation of $\Phi \vdash_{\Delta} \Psi$,
then Π^{π} is a valid derivation of $\Phi^{\pi} \vdash_{\Delta^{\pi}} \Psi^{\pi}$.

Theorem 2 If Π is a valid derivation of $\Phi \vdash_{\Delta} \Psi$ and $\Delta' \vdash \Delta\sigma$,
then $\Pi(\sigma, \Delta')$ is a valid derivation of $\Phi\sigma \vdash_{\Delta'} \Psi\sigma$.

$\Pi(\sigma, \Delta')$ is Π in which:

- each unknown X is replaced by $\sigma(X)$;
- each freshness context Δ is replaced by Δ'.

Properties of the sequent calculus (2)

For example, Π is the derivation of $a\#P \to P \supset P[a \mapsto T]$:

$$\dfrac{\dfrac{\overline{P \vdash_{a\#P} P}\ (\mathbf{Ax})}{P \vdash_{a\#P} P[a \mapsto T]}\ (\mathbf{StructR}) \quad (a\#P \vdash_{\textsf{SUB}} P = P[a \mapsto T])}{\vdash_{a\#P} P \supset P[a \mapsto T]}\ (\supset\mathbf{R})$$

Take $\sigma = [\mathsf{p}(c)/P, d/T]$ and $\Delta' = \emptyset$, then:

- $\Delta' \vdash \Delta\sigma$, i.e. $\emptyset \vdash a\#\mathsf{p}(c)$;
- $\Pi(\sigma, \Delta')$ is the following valid derivation of $\mathsf{p}(c) \supset \mathsf{p}(c)[a \mapsto d]$:

$$\dfrac{\dfrac{\overline{\mathsf{p}(c) \vdash_{\emptyset} \mathsf{p}(c)}\ (\mathbf{Ax})}{\mathsf{p}(c) \vdash_{\emptyset} \mathsf{p}(c)[a \mapsto d]}\ (\mathbf{StructR}) \quad (\emptyset \vdash_{\textsf{SUB}} \mathsf{p}(c) = \mathsf{p}(c)[a \mapsto d])}{\vdash_{\emptyset} \mathsf{p}(c) \supset \mathsf{p}(c)[a \mapsto d]}\ (\supset\mathbf{R})$$

Properties of the sequent calculus (3)

Theorem 3 [Cut elimination]
The $(\mathbf{Cut})$ rule is admissible in the system without it.

Corollary 4 The sequent calculus is **consistent**, i.e. $\vdash_{\triangle}$ can never be derived.

Axiomatisation of one-and-a-halfth-order logic

Theory FOL extends theory SUB with the following axioms:

$$P \supset Q \supset P \;=\; \top \qquad \neg\neg P \supset P \;=\; \top \qquad \textbf{(Props)}$$
$$(P \supset Q) \supset (Q \supset R) \supset (P \supset R) \;=\; \top \qquad \bot \supset P \;=\; \top$$

$$\forall[a]P \supset P[a \mapsto T] \;=\; \top \qquad \textbf{(Quants)}$$
$$\forall[a](P \wedge Q) \Leftrightarrow \forall[a]P \wedge \forall[a]Q \;=\; \top$$
$$a\#P \;\rightarrow\; \forall[a](P \supset Q) \Leftrightarrow P \supset \forall[a]Q \;=\; \top$$

$$T \approx T = \top \qquad U \approx T \wedge P[a \mapsto T] \supset P[a \mapsto U] = \top \qquad \textbf{(Eq)}$$

Axioms are all of the form $\phi = \top$, which intuitively means ‘ϕ is true’.

Note that this is a *finite* number of axioms.

Axiomatisation of one-and-a-halfth-order logic (2)

The **conjunctive form** $\Phi^\wedge$ of a predicate contexts Φ is Φ where we put $\wedge$ between its elements. Analogously, define its **disjunctive form** by putting $\vee$ between its elements. For example:

$$\emptyset^\wedge = \top \qquad \{\phi, \psi\}^\wedge = \phi \wedge \psi \qquad \emptyset^\vee = \bot \qquad \{\phi, \psi\}^\vee = \phi \vee \psi$$

Theorem 5 For all predicate contexts Φ, Ψ and freshness contexts Δ:

$$\Phi \vdash_\Delta \Psi \text{ is derivable} \quad \text{iff} \quad \Delta \vdash_{\mathsf{FOL}} \Phi^\wedge \supset \Psi^\vee = \top.$$

So sequent and equational derivability are equivalent.

Corollary 6 Theory FOL is consistent, i.e. $\Delta \vdash_{\mathsf{FOL}} \top = \bot$ does not hold.

Relation to First-order Logic

Call a term or a predicate context **ground** if it does not contain unknowns or explicit substitutions.

Call $\Phi \vdash \Psi$ a **first-order sequent**, when Φ and Ψ are ground predicate contexts.

Gentzen's sequent calculus for first-order logic:

$$\frac{}{\phi,\ \Phi \vdash \Psi,\ \phi}(\mathbf{Ax}) \qquad \frac{}{\bot,\ \Phi \vdash \Psi}(\bot\mathbf{L})$$

$$\frac{\Phi \vdash \Psi,\ \phi \quad \psi,\ \Phi \vdash \Psi}{\phi \supset \psi,\ \Phi \vdash \Psi}(\supset\mathbf{L}) \qquad \frac{\phi,\ \Phi \vdash \Psi,\ \psi}{\Phi \vdash \Psi,\ \phi \supset \psi}(\supset\mathbf{R})$$

$$\frac{\phi[\![a \mapsto t]\!],\ \Phi \vdash \Psi}{\forall a.\phi,\ \Phi \vdash \Psi}(\forall\mathbf{L}) \qquad \frac{\Phi \vdash \Psi,\ \phi}{\Phi \vdash \Psi,\ \forall a.\phi}(\forall\mathbf{R}) \quad (a \notin fn(\Phi, \Psi))$$

$$\frac{\phi[\![a \mapsto t']\!],\ \Phi \vdash \Psi}{t' \approx t,\ \phi[\![a \mapsto t]\!],\ \Phi \vdash \Psi}(\approx \mathbf{L}) \qquad \frac{}{\Phi \vdash \Psi,\ t \approx t}(\approx \mathbf{R})$$

Relation to First-order Logic (2)

Note that:

- we write $\forall a.\phi$ for $\forall[a]\phi$;
- $[\![a \mapsto t]\!]$ is capture-avoiding substitution;
- $a \notin fn(\phi)$ is 'a does not occur in the free names of ϕ';
- we take predicates up to α-equivalence.

Theorem 7 $\Phi \vdash \Psi$ is derivable in the sequent calculus for first-order logic, iff $\Phi \vdash_{\emptyset} \Psi$ is derivable in the sequent calculus for one-and-a-halfth-order logic.

So on ground terms, one-and-a-halfth-order logic *is* first-order logic.

Semantics

For closed terms t, its **ground form** $t[\![]\!]$ is t in which each explicit substitution $v[a \mapsto u]$ is replaced by $v[\![a \mapsto u]\!]$ bottom-up in the syntax.

Theorem 8 For closed terms t, $\vdash_{\mathsf{SUB}} t = t[\![]\!]$

Call a substitution σ **closing for a term** t if $t\sigma$ is closed.

A term-in-context $\Delta \to \phi$ is **valid** iff for all closing substitutions σ (for ϕ) for which $\vdash \Delta\sigma$ holds, $\phi\sigma[\![]\!]$ is valid in the semantics of first-order logic.

The sequent calculus for one-and-a-halfth-order logic is **sound** for this semantics:

Theorem 9 If $\vdash_{\Delta} \phi$ is derivable then $\Delta \to \phi$ is valid.

Conclusions

Using nominal terms, we can:

- *accurately* represent systems with binding: e.g. explicit substitution and first-order logic;
- specify *novel* systems with their own mathematical interest: e.g. one-and-a-halfth-order logic.

One-and-a-halfth-order logic:

- makes meta-level concepts of first-order logic *explicit*;
- has a sequent calculus with *syntax-directed* rules;
- has a *semantics* in first-order logic;
- has a *finite* equational axiomatisation;
- is the *result* of axiomatising first-order logic in nominal algebra.

Related work

Second-order logic (SOL):

- In this logic we can quantify over predicates *anywhere*, which makes it more expressive than one-and-a-halfh-order logic.
- On the other hand, we can easily extend theory FOL with *one* axiom to express the principle of induction on natural numbers:

$$P[a \mapsto 0] \wedge \forall[a](P \supset P[a \mapsto succ(a)]) \supset \forall[a]P \ = \ \top.$$

Higher-order logic (HOL) is type raising, while one-and-a-halfth-order logic is *not*: $P[a \mapsto t]$ corresponds to $f(t)$ in HOL, where $f : \mathbb{T} \to \mathbb{P}$; $P[a \mapsto t][a' \mapsto t']$ corresponds to $f'(t)(t')$ where $f' : \mathbb{T} \to \mathbb{T} \to \mathbb{P}$, and so on...

One-and-a-halfth-order logic is not a subset of SOL or HOL because of freshnesses.

Future work

- Completeness of the sequent calculus with respect to the semantics.
- Let unknowns range over *sequent derivations*, and establish a Curry-Howard correspondence (term-in-contexts as types, derivations as terms).
- Two-and-a-halfth-order logic (where you can abstract X)?
- Implementation and automation?

Current status

- M.J. Gabbay, A.H.J. Mathijssen, Nominal Algebra, submitted CSL'06.
- M.J. Gabbay, A.H.J. Mathijssen, Capture-avoiding Substitution as a Nominal Algebra, submitted ICTAC'06.
- M.J. Gabbay, A.H.J. Mathijssen, One-and-a-halfth-order Logic, PPDP'06.

Just to scare you

$$
\begin{array}{cl}
\hline
P[b \mapsto c][a \mapsto c] \vdash_{c\#P} P[b \mapsto c][a \mapsto c] & (\mathbf{Ax}) \\
\hline
\forall[a]P[b \mapsto c] \vdash_{c\#P} P[b \mapsto c][a \mapsto c] & (\forall\mathbf{L}) \\
\hline
(\forall[a]P)[b \mapsto c] \vdash_{c\#P} P[b \mapsto a][a \mapsto c] & (\mathbf{StructL}) \quad (1.) \\
\hline
\forall[b]\forall[a]P \vdash_{c\#P} P[b \mapsto c][a \mapsto c] & (\forall\mathbf{L}) \\
\hline
\forall[b]\forall[a]P \vdash_{c\#P} \forall[c]P[b \mapsto c][a \mapsto c] & (\forall\mathbf{R}) \quad (2.) \\
\hline
\forall[b]\forall[a]P \vdash_{c\#P} \forall[a]P[b \mapsto a] & (\mathbf{StructR}) \quad (3.) \\
\hline
\forall[b]\forall[a]P \vdash_{\emptyset} \forall[a]P[b \mapsto a] & (\mathbf{Fresh}) \quad (4.)
\end{array}
$$

Side-conditions:

1. $c\#P \vdash_{\mathsf{SUB}} \forall[a]P[b \mapsto c] = (\forall[a]P)[b \mapsto c]$
2. $c\#P \vdash c\#\forall[b]\forall[a]P$
3. $c\#P \vdash_{\mathsf{SUB}} \forall[c]P[b \mapsto c][a \mapsto c] = \forall[a]P[b \mapsto a]$
4. $c \notin \forall[b]\forall[a]P,\ \forall[a]P[b \mapsto a]$