One-and-a-halfth-order Logic

Aad Mathijssen　　Murdoch J. Gabbay

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Introduction

Consider the following valid assertions in first-order logic:

- $\phi \supset (\psi \supset \phi)$
- if $a \notin fn(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin fn(\phi)$ then $\phi \supset (\phi[\![a \mapsto t]\!])$

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These are *not valid syntax* in first-order logic, because of *meta-level concepts*:

- meta-variables *varying* over syntax: ϕ, ψ, a, t
- properties of syntax:
  - freshness assumptions: $a \notin fn(\phi)$
  - capture-avoiding substitution: $\phi[\![a \mapsto t]\!]$

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Is there a logic in which the above assertions are valid syntax?

Introduction (2)

Consider the following (sequent) derivations:

$$\dfrac{\dfrac{\dfrac{}{\phi, \psi \vdash \phi}\,(\mathbf{Ax})}{\phi \vdash \psi \supset \phi}\,(\supset\mathbf{R})}{\vdash \phi \supset (\psi \supset \phi)}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{\dfrac{}{\bot, \bot \vdash \bot}\,(\mathbf{Ax})}{\bot \vdash \bot \supset \bot}\,(\supset\mathbf{R})}{\vdash \bot \supset (\bot \supset \bot)}\,(\supset\mathbf{R})$$

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The left one is not a derivation, it is a *schema* of derivations.

The right one is a derivation, it is an *instance* of the schema on the left.

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The left one is not a derivation, it is a *schema* of derivations.

The right one is a derivation, it is an *instance* of the schema on the left.

Is there a logic in which the derivation on the left is a derivation too?

Introduction (3)

One-and-a-halfth-order logic makes meta-level concepts *explicit*.

The following *judgements* are valid in one-and-a-halfth-order logic:

- $P \supset (Q \supset P) = \top$
- $a\#P \rightarrow P \supset \forall[a]P = \top$
- $a\#P \rightarrow P \supset (P[a \mapsto T]) = \top$

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No meta-level concepts:

- P, Q and T are *unknowns*, representing meta-level variables
- a is an *atom*, representing an object-level variable
- $a\#P$ is a *freshness*, representing a is fresh for P
- $P[a \mapsto T]$ is an *explicit substitution*, repr. capture-avoiding substitution

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The following (sequent) derivation is valid in one-and-a-halfth-order logic:

$$\dfrac{\dfrac{\overline{P \vdash_{a\#P} P}\ (\mathbf{Ax})}{P \vdash_{a\#P} \forall[a]P}\ (\forall\mathbf{R})\quad (a\#P \vdash a\#P)}{\vdash_{a\#P} P \supset \forall[a]P}\ (\supset\mathbf{R})$$

Side condition $a\#P \vdash a\#P$:
freshness $a\#P$ is *derivable* from the *assumption* $a\#P$.

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The following (sequent) derivation is valid in one-and-a-halfth-order logic:

$$\dfrac{\dfrac{\overline{P \vdash_{a\#P} P}\,(\mathbf{Ax})}{P \vdash_{a\#P} P[a \mapsto T]}\,(\mathbf{StructR}) \quad (a\#P \vdash_{\mathsf{SUB}} P = P[a \mapsto T])}{\vdash_{a\#P} P \supset (P[a \mapsto T])}\,(\supset\mathbf{R})$$

Side condition $a\#P \vdash_{\mathsf{SUB}} P = P[a \mapsto T]$:
equality $P = P[a \mapsto T]$ is *derivable* from the *assumption* $a\#P$ in *theory* SUB.

The rule $(\mathbf{StructR})$ lets us replace the right-hand side $P[a \mapsto T]$ of the equality assertion by its left-hand side P.

Overview

- Nominal Algebra:
  - Signature, axioms and theories
  - Equational theory of one-and-a-halfth order logic
  - Equational proof system
- Sequent calculus for one-and-a-halfth-order logic
- Relation to first-order logic
- Conclusions, related and future work

Nominal Algebra...

...is a theory of algebraic equality on *nominal terms*.

...has built-in support for *binding* and *freshnesses*.

...is *first-order*, not higher-order.

...allows for *direct* and *natural* representation of existing systems with binding.

...also allows for *novel* systems like one-and-a-halfth-order logic.

Signature

δ ranges over **base sorts**.

$\mathbb{A}$ ranges over **atomic sorts**.

Sorts τ:

$$\tau ::= \delta \mid \mathbb{A} \mid [\mathbb{A}]\tau$$

Term-formers f_ρ have an associated **arity** $\rho = (\tau_1, \ldots, \tau_n)\tau$.
$\mathsf{f} : \rho$ means 'f with arity ρ'.

A **signature** $\Sigma = (D, A, F)$ where D, A and F are finite sets of base sorts, atomic sorts and term-formers.

Signature (2)

Atoms $a, b, c, \ldots$ have sort $\mathbb{A}$; they represent *object-level* variable symbols.

Unknowns $X, Y, Z, \ldots$ have sort τ; they represent *meta-level* variable symbols.

A **permutation** π of atoms is a total bijection $\mathbb{A} \rightarrow \mathbb{A}$ with finite support:
$\pi(a) \neq a$ for a finite number of a's and $\pi(a) = a$ for all others.

We call $\pi \cdot X$ a **moderated unknown**.
This represents the permutation of atoms π acting on an unknown term.

Terms t, subscripts indicate sorting rules:

$$t \ ::= a_{\mathbb{A}} \mid (\pi \cdot X_\tau)_\tau \mid [a_{\mathbb{A}}]t_\tau \mid (\mathsf{f}_{(\tau_1,\ldots,\tau_n)\tau}(t^1_{\tau_1}, \ldots, t^n_{\tau_n}))_\tau$$

Signature (3)

Signature for one-and-a-halfth-order logic:

- Base sorts $\mathbb{F}$ for 'formulae' and $\mathbb{T}$ for 'terms'; atomic sort $\mathbb{A}$;
- Term-formers:
  - $\bot : ()\mathbb{F}$ represents *falsity*;
  - $\supset : (\mathbb{F}, \mathbb{F})\mathbb{F}$ represents *implication*, write $\phi \supset \psi$ for $\supset(\phi, \psi)$;
  - $\forall : ([\mathbb{A}]\mathbb{F})\mathbb{F}$ represents *universal quantification*, write $\forall[a]\phi$ for $\forall([a]\phi)$;
  - $\approx : (\mathbb{T}, \mathbb{T})\mathbb{F}$ represents *object-level equality*, write $t \approx u$ for $\approx(t, u)$;
  - $\mathsf{var} : (\mathbb{A})\mathbb{T}$ is *variable casting*, forced upon us by the sort system;
  - $\mathsf{sub} : ([\mathbb{A}]\tau, \mathbb{T})\tau$, where $\tau \in \{\mathbb{F}, \mathbb{T}, [\mathbb{A}]\mathbb{F}\}$, is *explicit substitution*, write $t[a \mapsto u]$ for $\mathsf{sub}([a]t, u)$;
  - $\mathsf{p}_1, \ldots, \mathsf{p}_n : (\mathbb{T}, \ldots, \mathbb{T})\mathbb{F}$ are *object-level predicate term-formers*;
  - $\mathsf{f}_1, \ldots, \mathsf{f}_m : (\mathbb{T}, \ldots, \mathbb{T})\mathbb{T}$ are *object-level term-formers*.

Signature (4)

Sugar:

$$\top \text{ is } \bot \supset \bot \qquad \neg\phi \text{ is } \phi \supset \bot \qquad \phi \wedge \psi \text{ is } \neg(\phi \supset \neg\psi)$$
$$\phi \vee \psi \text{ is } \neg\phi \supset \psi \qquad \phi \Leftrightarrow \psi \text{ is } (\phi \supset \psi) \wedge (\psi \supset \phi) \qquad \exists[a]\phi \text{ is } \neg\forall[a]\phi$$

Descending order of operator precedence:

$$\_[\_ \mapsto \_],\ \approx,\ \{\neg, \forall, \exists\},\ \{\wedge, \vee\},\ \supset,\ \Leftrightarrow$$

$\wedge$, $\vee$ and $\supset$ associate to the right.

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Example terms of sort $\mathbb{F}$:

$$P \supset Q \supset P \qquad P \supset \forall[a]P \qquad P \supset P[a \mapsto T]$$

P, Q are unknowns of sort $\mathbb{F}$, T is an unknown of sort $\mathbb{T}$, a is an atom of sort $\mathbb{A}$.

Assertions and judgements

Freshness (assertions) $a\#t$, which means ‘a is fresh for t.
If t is an unknown X, the freshness is called **primitive**.

Equality (assertions) $t = u$, where t and u are of the same sort.

Write Δ for a set of *primitive* freshnesses and call it a **freshness context**.
We may leave out set brackets, writing $a\#X, b\#Y$ instead of $\{a\#X, b\#Y\}$.

We call $\Delta \to A$ a **judgement** where A is an assertion ($a\#t$ or $t = u$).
We may leave out $\Delta \to$ if Δ is empty ($\emptyset$).

Assertions and judgements (2)

Example equality judgements:

- $\emptyset \to P \supset Q \supset P = \top$, or just $P \supset Q \supset P = \top$
- $\{a\#P\} \to P \supset \forall[a]P = \top$, or just $a\#P \to P \supset \forall[a]P = \top$
- $\{a\#P\} \to P \supset P[a \mapsto T] = \top$, or just $a\#P \to P \supset P[a \mapsto T] = \top$

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When are these *valid*?

Axioms and theories

We allow equality judgements $\Delta \to t = u$ with finite Δ as **axioms**.

A **theory** $\mathsf{T} = (\Sigma, Ax)$ where:

- Σ is a signature;
- Ax is a possibly infinite set of axioms.

Axioms and theories (2)

- CORE: a theory of α-conversion
- SUB: a theory of explicit substitution
- FOL: a theory of one-and-a-halfth-order logic (watch the name)

Relation between the theories:

- Signature is the same (previously introduced)
- Axioms of smaller theories are contained in bigger ones according to the following relation:

$$\mathsf{CORE} \subset \mathsf{SUB} \subset \mathsf{FOL}$$

Axioms and theories (3)

Axioms of CORE: none!

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Axioms of SUB:

$$
\begin{array}{rr}
(\mathbf{f}\mapsto) & \mathsf{f}(X_1,\ldots,X_n)[a\mapsto T] = \mathsf{f}(X_1[a\mapsto T],\ldots,X_n[a\mapsto T]) \\
(\mathbf{abs}\mapsto) & b\#T \to ([b]X)[a\mapsto T] = [b](X[a\mapsto T]) \\
(\mathbf{var}\mapsto) & \mathsf{var}(a)[a\mapsto T] = T \\
(\#\mapsto) & a\#X \to X[a\mapsto T] = X \\
(\mathbf{ren}\mapsto) & b\#X \to X[a\mapsto \mathsf{var}(b)] = (b\ a)\cdot X
\end{array}
$$

f ranges over all term-formers excluding var, but including sub.
a and b are distinct atoms.
T is an unknown of sort $\mathbb{T}$, $X, X_1, \ldots, X_n$ are unknowns of appropriate sorts.

Note that this is a *finite* number of axioms.

Axioms and theories (4)

Axioms of FOL: axioms of SUB extended with

$$P \supset Q \supset P \;=\; \top \qquad \neg\neg P \supset P \;=\; \top \qquad \textbf{(Props)}$$

$$(P \supset Q) \supset (Q \supset R) \supset (P \supset R) \;=\; \top \qquad \bot \supset P \;=\; \top$$

$$\forall[a]P \supset P[a \mapsto T] \;=\; \top \qquad \textbf{(Quants)}$$

$$\forall[a](P \wedge Q) \Leftrightarrow \forall[a]P \wedge \forall[a]Q \;=\; \top$$

$$a\#P \;\rightarrow\; \forall[a](P \supset Q) \Leftrightarrow P \supset \forall[a]Q \;=\; \top$$

$$T \approx T = \top \qquad U \approx T \wedge P[a \mapsto T] \supset P[a \mapsto U] = \top \qquad \textbf{(Eq)}$$

T, U are unknowns of sort $\mathbb{T}$, P, Q, R are unknowns of sort $\mathbb{F}$.
Axioms are all of the form $\phi = \top$, which intuitively means 'ϕ is true'.

Note that this is a *finite* number of axioms.

Validity in theory FOL

Example equality judgements:

- $P \supset Q \supset P = \top$
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How can we show that these are valid in theory FOL?

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How can we show that these are valid in theory FOL?

Semantics of Nominal Algebra: not treated here.

Sound and complete *proof system* for Nominal Algebra: treated here.

Derivability of freshnesses

$$\frac{}{a\#b}\,(\#\mathbf{ab}) \quad \frac{a\#t_1 \ \cdots \ a\#t_n}{a\#\mathsf{f}(t_1,\ldots,t_n)}\,(\#\mathbf{f}) \quad \frac{\pi^{\text{-}1}(a)\#X}{a\#\pi\cdot X}\,(\#\mathbf{X})$$

$$\frac{}{a\#[a]t}\,(\#[]\mathbf{a}) \quad \frac{a\#t}{a\#[b]t}\,(\#[]\mathbf{b})$$

a and b range over distinct atoms.

Write $\Delta \vdash a\#t$ when there exists a derivation of $a\#t$ using the elements of Δ as assumptions. Say that $a\#t$ **is derivable from** Δ.

A freshness judgement $\Delta \to a\#t$ is derivable when $\Delta \vdash a\#t$.

Derivability of equalities

$$\frac{}{t = t}\,(\mathbf{refl}) \qquad \frac{t = u}{u = t}\,(\mathbf{symm}) \qquad \frac{t = u \quad u = v}{t = v}\,(\mathbf{tran})$$

$$\frac{t = u}{C[t] = C[u]}\,(\mathbf{cong}) \qquad \frac{a\#t \quad b\#t}{(a\ b)\cdot t = t}\,(\mathbf{perm})$$

$$\frac{\Delta^\pi \sigma}{t^\pi\sigma = u^\pi\sigma}\,(\mathbf{ax_A})\ A \equiv \Delta \to t = u \qquad \frac{\begin{matrix}[a\#X_1, \ldots, a\#X_n] \quad \Delta \\ \vdots \\ t = u\end{matrix}}{t = u}\,(\mathbf{fr}) \quad (a \notin t, u, \Delta)$$

Here A is an axiom, and we call $C[\_]$ a **context**.

Write $\Delta \vdash_{\mathsf{T}} t = u$ when $t = u$ **is derivable from** Δ using axioms from T only.

$\Delta \to t = u$ is derivable in theory T when $\Delta \vdash_{\mathsf{T}} t = u$.

Derivability of equalities (2)

Write $\equiv$ for **syntactic identity**.

Define **permutation actions** on terms $\pi \cdot t,\ t^{\pi}$:

$$\pi \cdot a \equiv \pi(a) \qquad \pi \cdot (\pi' \cdot X) \equiv (\pi \circ \pi') \cdot X$$
$$\pi \cdot [a]t \equiv [\pi(a)](\pi \cdot t) \qquad \pi \cdot \mathsf{f}(t_1, \ldots, t_n) \equiv \mathsf{f}(\pi \cdot t_1, \ldots, \pi \cdot t_n)$$

$$a^{\pi} \equiv \pi(a) \qquad (\pi' \cdot X)^{\pi} \equiv (\pi \circ \pi' \circ \pi^{-1}) \cdot X$$
$$([a]t)^{\pi} \equiv [\pi(a)](t^{\pi}) \qquad \mathsf{f}(t_1, \ldots, t_n)^{\pi} \equiv \mathsf{f}({t_1}^{\pi}, \ldots, {t_n}^{\pi})$$

A **substitution** σ is an assignment of unknowns to terms of the same sort.
Define a **substitution action** on terms $t\sigma$:

$$a\sigma \equiv a \qquad (\pi \cdot X)\sigma \equiv \pi \cdot \sigma(X)$$
$$([a]t)\sigma \equiv [a]t\sigma \qquad \mathsf{f}(t_1, \ldots, t_n)\sigma \equiv \mathsf{f}(t_1\sigma, \ldots, t_n\sigma)$$

Derivability of equalities (3)

Derivable equality judgements in FOL:

- $P \supset Q \supset P = \top$, i.e. $\vdash_{\mathsf{FOL}} P \supset Q \supset P = \top$.
- $a \# P \to P \supset \forall[a]P = \top$, i.e. $a \# P \vdash_{\mathsf{FOL}} P \supset \forall[a]P = \top$
- $a \# P \to P \supset P[a \mapsto T] = \top$, i.e. $a \# P \vdash_{\mathsf{FOL}} P \supset P[a \mapsto T] = \top$

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This concludes the treatment of the *equational* proof system for FOL.
Let's have a look at a *sequent calculus* for FOL.

A sequent calculus for FOL

Sequent calculi are often more effective in proving assertions than equational proof systems.

We may call terms of sort $\mathbb{F}$ **formulae**, and denote them by ϕ and ψ.

Let **(formula) contexts** Φ, Ψ be finite sets of formulae.
We may write ϕ for $\{\phi\}$, ϕ, Φ for $\{\phi\} \cup \Phi$, and Φ, Φ' for $\Phi \cup \Phi'$.

A **sequent** is a triple $\Phi \vdash_{\scriptscriptstyle\triangle} \Psi$.
We may omit empty formula contexts, e.g. writing $\vdash_{\scriptscriptstyle\triangle}$ for $\emptyset \vdash_{\scriptscriptstyle\triangle} \emptyset$.

Define derivability on sequents...

A sequent calculus for FOL (2)

Rules resembling Gentzen's sequent calculus for first-order logic:

$$\frac{}{\phi,\ \Phi \vdash_{\Delta} \Psi,\ \phi}\,(\mathbf{Ax}) \qquad \frac{}{\bot,\ \Phi \vdash_{\Delta} \Psi}\,(\bot\mathbf{L})$$

$$\frac{\Phi \vdash_{\Delta} \Psi,\ \phi \quad \psi,\ \Phi \vdash_{\Delta} \Psi}{\phi \supset \psi,\ \Phi \vdash_{\Delta} \Psi}\,(\supset\mathbf{L}) \qquad \frac{\phi,\ \Phi \vdash_{\Delta} \Psi,\ \psi}{\Phi \vdash_{\Delta} \Psi,\ \phi \supset \psi}\,(\supset\mathbf{R})$$

$$\frac{\phi[a \mapsto t],\ \Phi \vdash_{\Delta} \Psi}{\forall[a]\phi,\ \Phi \vdash_{\Delta} \Psi}\,(\forall\mathbf{L}) \qquad \frac{\Phi \vdash_{\Delta} \Psi,\ \psi}{\Phi \vdash_{\Delta} \Psi,\ \forall[a]\psi}\,(\forall\mathbf{R}) \quad (\Delta \vdash a\#\Phi, \Psi)$$

$$\frac{\phi[a \mapsto t'],\ \Phi \vdash_{\Delta} \Psi}{t' \approx t,\ \phi[a \mapsto t],\ \Phi \vdash_{\Delta} \Psi}\,(\approx\mathbf{L}) \qquad \frac{}{\Phi \vdash_{\Delta} \Psi,\ t \approx t}\,(\approx\mathbf{R})$$

These are *schemas*: a ranges over atoms, t, t' ranges over terms of sort $\mathbb{T}$, ϕ, ψ range over formulae, and Φ, Ψ range over formula contexts.

A sequent calculus for FOL (3)

Other rules:

$$\frac{\phi', \Phi \vdash_{\Delta} \Psi}{\phi, \Phi \vdash_{\Delta} \Psi}\,(\mathbf{StructL}) \quad (\Delta \vdash_{\mathsf{SUB}} \phi' = \phi)$$

$$\frac{\Phi \vdash_{\Delta} \Psi, \psi'}{\Phi \vdash_{\Delta} \Psi, \psi}\,(\mathbf{StructR}) \quad (\Delta \vdash_{\mathsf{SUB}} \psi' = \psi)$$

$$\frac{\Phi \vdash_{\Delta \cup \{a \# X_1, \ldots, X_n\}} \Psi}{\Phi \vdash_{\Delta} \Psi}\,(\mathbf{Fresh}) \quad (a \notin \Phi, \Psi, \Delta)$$

$$\frac{\Phi \vdash_{\Delta} \Psi, \phi \quad \phi', \Phi \vdash_{\Delta} \Psi}{\Phi \vdash_{\Delta} \Psi}\,(\mathbf{Cut}) \quad (\Delta \vdash_{\mathsf{SUB}} \phi = \phi')$$

Properties of the sequent calculus

For $\Phi \equiv \{\phi_1, \ldots, \phi_n\}$, define its **conjunctive form** $\Phi^\wedge$ to be $\phi_1 \wedge \cdots \wedge \phi_n$ when $n > 0$, and $\top$ when $n = 0$. Analogously, define the **disjunctive form** $\Phi^\vee$ to be $\phi_1 \vee \cdots \vee \phi_n$ when $n > 0$, and $\perp$ when $n = 0$.

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Theorem 1 For all FOL contexts Φ, Ψ and freshness contexts Δ:

$$\Phi \vdash_{\Delta} \Psi \text{ is derivable} \quad \text{iff} \quad \Delta \vdash_{\mathsf{FOL}} \Phi^\wedge \supset \Psi^\vee = \top.$$

So equational and sequent derivability are equivalent.

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So equational and sequent derivability are equivalent.

Theorem 2 If Π is a derivation of $\Phi \vdash_{\Delta} \Psi$ and $\Delta' \vdash \Delta^\pi\sigma$, then there exists a derivation Π' of $\Phi^\pi\sigma \vdash_{\Delta'} \Psi^\pi\sigma$, which is Π in which atoms are *permuted*, unknowns are *instantiated*, and freshness contexts are *replaced*.

Properties of the sequent calculus (2)

Theorem 3 [Cut elimination]
The $(\mathbf{Cut})$ rule is admissible in the system without it.

Properties of the sequent calculus (2)

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The (**Cut**) rule is admissible in the system without it.

Corollary 4 The sequent calculus and the equational proof systems for FOL are both **consistent**, i.e. for any freshness context Δ:

- $\vdash_{\Delta}$ cannot be derived;
- $\Delta \to \top = \bot$ cannot be derived in FOL.

Relation to First-order Logic

Call a term **ground** if it does not contain unknowns or explicit substitutions. From now on we only consider terms and formula contexts on ground terms.

A **first-order sequent** is a pair $\Phi \vdash \Psi$.

Genzten's sequent calculus for first-order logic:

$$\frac{}{\phi,\ \Phi \vdash \Psi,\ \phi}\,(\mathbf{Ax}) \qquad \frac{}{\bot,\ \Phi \vdash \Psi}\,(\bot\mathbf{L})$$

$$\frac{\Phi \vdash \Psi,\ \phi \quad \psi,\ \Phi \vdash \Psi}{\phi \supset \psi,\ \Phi \vdash \Psi}\,(\supset\mathbf{L}) \qquad \frac{\phi,\ \Phi \vdash \Psi,\ \psi}{\Phi \vdash \Psi,\ \phi \supset \psi}\,(\supset\mathbf{R})$$

$$\frac{\phi[\![a \mapsto t]\!],\ \Phi \vdash \Psi}{\forall a.\phi,\ \Phi \vdash \Psi}\,(\forall\mathbf{L}) \qquad \frac{\Phi \vdash \Psi,\ \phi}{\Phi \vdash \Psi,\ \forall a.\phi}\,(\forall\mathbf{R}) \quad (a \notin fn(\Phi, \Psi))$$

$$\frac{\phi[\![a \mapsto t']\!],\ \Phi \vdash \Psi}{t' \approx t,\ \phi[\![a \mapsto t]\!],\ \Phi \vdash \Psi}\,(\approx\mathbf{L}) \qquad \frac{}{\Phi \vdash \Psi,\ t \approx t}\,(\approx\mathbf{R})$$

Relation to First-order Logic (2)

Note that:

- We write $\forall a.\phi$ for $\forall[a]\phi$.
- $[\![a \mapsto t]\!]$ is capture-avoiding substitution.
- $a \notin fn(\phi)$ is 'a does not occur in the free names of ϕ'.
- We take formulae up to α-equivalence, e.g. suppose $\mathsf{p} : (\mathbb{T})\mathbb{F}$ is an atomic predicate term-former, then $\forall a.\mathsf{p}(a) \vdash \forall b.\mathsf{p}(b)$ follows directly by $(\mathbf{Ax})$ since $\forall a.\mathsf{p}(a) =_\alpha \forall b.\mathsf{p}(b)$.

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Theorem 5 $\Phi \vdash \Psi$ is derivable in the sequent calculus for first-order logic, if and only if $\Phi \vdash_\emptyset \Psi$ is derivable in the sequent calculus for FOL.

So on ground terms, one-and-a-halfth-order logic *is* first-order logic.

Conclusions

Nominal algebra:

- is a system in which we can *accurately* represent systems with binding: e.g. explicit substitution and first-order logic;
- allows for *novel* systems with their own mathematical interest: e.g. one-and-a-halfth-order logic.

One-and-a-halfth-order logic:

- is the *result* of axiomatising first-order logic in Nominal algebra;
- makes meta-level concepts of first-order logic *explicit*;
- has a *finite* equational axiomatisation;
- has a sequent calculus with *syntax-directed* rules;
- has a *semantics* in first-order logic on ground terms.

Related work

Second-order logic:

- In this logic we can quantify over predicates *anywhere*, which makes it more expressive than one-and-a-halfh-order logic.
- Theory FOL does have a second-order flavour. It can easily be extended with one axiom that expresses the principle of induction on natural numbers:

$$P[a \mapsto 0] \land \forall[a](P \supset P[a \mapsto succ(\mathsf{var}(a))]) \supset \forall[a]P \ = \ \top.$$

Higher-order logic (HOL):

- is type raising, while one-and-a-halfth-order logic is *not*: $P[a \mapsto t]$ corresponds to $f(t)$ in HOL, where $f : \mathbb{T} \to \mathbb{F}$; $P[a \mapsto t][a' \mapsto t']$ corresponds to $f'(t)(t')$ where $f' : \mathbb{T} \to \mathbb{T} \to \mathbb{F}$, and so on...
- One-and-a-halfth-order logic is not a subset of HOL because of freshnesses.

Future work

- Concrete semantics for one-and-a-halfth-order logic on non-ground terms.
- Two-and-a-halfth-order logic (where you can abstract X)?
- Implementation and automation?

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Current status

- M.J. Gabbay, A.H.J. Mathijssen, Nominal Algebra, submitted CSL'06.
- M.J. Gabbay, A.H.J. Mathijssen, Capture-avoiding Substitution as a Nominal Algebra, submitted ICTAC'06.
- M.J. Gabbay, A.H.J. Mathijssen, One-and-a-halfth-order Logic, submitted PPDP'06.

Just to scare you

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P[b \mapsto \mathsf{var}(c)][a \mapsto \mathsf{var}(c)] \vdash_{c\#P} P[b \mapsto \mathsf{var}(c)][a \mapsto \mathsf{var}(c)]}\,(\mathbf{Ax})}{\forall[a](P[b \mapsto \mathsf{var}(c)]) \vdash_{c\#P} P[b \mapsto \mathsf{var}(c)][a \mapsto \mathsf{var}(c)]}\,(\forall\mathbf{L})}{(\forall[a]P)[b \mapsto \mathsf{var}(c)] \vdash_{c\#P} P[b \mapsto \mathsf{var}(a)][a \mapsto \mathsf{var}(c)]}\,(\mathbf{StructL})\quad(1.)}{\forall[b]\forall[a]P \vdash_{c\#P} P[b \mapsto \mathsf{var}(c)][a \mapsto \mathsf{var}(c)]}\,(\forall\mathbf{L})}{\forall[b]\forall[a]P \vdash_{c\#P} \forall[c](P[b \mapsto \mathsf{var}(c)][a \mapsto \mathsf{var}(c)])}\,(\forall\mathbf{R})\quad(2.)}{\dfrac{\forall[b]\forall[a]P \vdash_{c\#P} \forall[a](P[b \mapsto \mathsf{var}(a)])}{\forall[b]\forall[a]P \vdash_{\emptyset} \forall[a](P[b \mapsto \mathsf{var}(a)])}\,(\mathbf{Fresh})\quad(4.)}\,(\mathbf{StructR})\quad(3.)$$

Side-conditions:

1. $c\#P \vdash_{\mathsf{SUB}} \forall[a](P[b \mapsto \mathsf{var}(c)]) = (\forall[a]P)[b \mapsto \mathsf{var}(c)]$
2. $c\#P \vdash c\#\forall[b]\forall[a]P$
3. $c\#P \vdash_{\mathsf{SUB}} \forall[c](P[b \mapsto \mathsf{var}(c)][a \mapsto \mathsf{var}(c)]) = \forall[a](P[b \mapsto \mathsf{var}(a)])$
4. $c \notin \forall[b]\forall[a]P,\ \forall[a](P[b \mapsto \mathsf{var}(a)])$