One-and-a-halfth-order Logic

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Motivation

Consider the following valid assertions in first-order logic:

- $\phi \supset \psi \supset \phi$
- if $a \notin fn(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin fn(\phi)$ then $\phi \supset \phi[\![a \mapsto t]\!]$
- if $b \notin fn(\phi)$ then $\forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]$

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These are not valid syntax in first-order logic.
This is because of meta-level concepts:

- meta-variables *varying* over syntax: ϕ, ψ, a, b, t
- properties of syntax: $a \notin fn(\phi)$, $\phi[\![a \mapsto t]\!]$, α-equivalence

Motivation (2)

Consider the following derivations in Gentzen's sequent calculus:

$$\dfrac{\dfrac{\dfrac{}{\psi, \phi \vdash \phi}\ (\mathbf{Ax})}{\phi \vdash \psi \supset \phi}\ (\supset\mathbf{R})}{\vdash \phi \supset \psi \supset \phi}\ (\supset\mathbf{R})$$

$$\dfrac{\dfrac{\dfrac{}{\mathrm{p}(d), \mathrm{p}(c) \vdash \mathrm{p}(c)}\ (\mathbf{Ax})}{\mathrm{p}(c) \vdash \mathrm{p}(d) \supset \mathrm{p}(c)}\ (\supset\mathbf{R})}{\vdash \mathrm{p}(c) \supset \mathrm{p}(d) \supset \mathrm{p}(c)}\ (\supset\mathbf{R})$$

And for $b \notin fn(\phi)$:

$$\dfrac{\dfrac{}{\forall a.\phi \vdash \forall b.\phi[\![a \mapsto b]\!]}\ (\mathbf{Ax})}{\vdash \forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]}\ (\supset\mathbf{R})$$

$$\dfrac{\dfrac{}{\forall c.\mathrm{p}(c) \vdash \forall d.\mathrm{p}(d)}\ (\mathbf{Ax})}{\vdash \forall c.\mathrm{p}(c) \supset \forall d.\mathrm{p}(d)}\ (\supset\mathbf{R})$$

Motivation (2)

Consider the following derivations in Gentzen's sequent calculus:

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And for $b \notin fn(\phi)$:

$$\dfrac{\dfrac{}{\forall a.\phi \vdash \forall b.\phi[\![a \mapsto b]\!]}\,(\mathbf{Ax})}{\vdash \forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{}{\forall c.\mathsf{p}(c) \vdash \forall d.\mathsf{p}(d)}\,(\mathbf{Ax})}{\vdash \forall c.\mathsf{p}(c) \supset \forall d.\mathsf{p}(d)}\,(\supset\mathbf{R})$$

The left ones are not derivations, they are *schemas* of derivations.
The right ones might be derivations; they *instances* of the schemas.

Motivation (3)

Questions:

- Is there a logic in which these schematic assertions and derivations are valid syntax too?

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- First-order logic and its proof systems formalise reasoning.
  But also a lot of reasoning is about first-order logic.
  So why shouldn't that be formalised?

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Questions:

- Is there a logic in which these schematic assertions and derivations are valid syntax too?
- First-order logic and its proof systems formalise reasoning. But also a lot of reasoning is about first-order logic. So why shouldn't that be formalised?

One-and-a-halfth-order logic tries to address this by formalising:

- meta-variables (ϕ, ψ, a, b, t)
- properties of syntax ($a \notin fn(\phi)$, $\phi[\![a \mapsto t]\!]$, α-equivalence)

Overview

- Definition of One-and-a-halfth-order Logic
  - Introduction
  - Formal syntax
  - Derivability
- Properties of One-and-a-halfth-order Logic
  - Proof-theoretical properties
  - Equational axiomatisation
  - Relation to first-order logic
  - Semantics
- Conclusions, related and future work

Introduction

In the syntax of one-and-a-halfth-order logic:

- *Unknowns* P, Q and T represent meta-variables ϕ, ψ and t.
- *Atoms* a and b represent meta-variables a and b.
- *Freshness* $a\#P$ represents $a \notin fn(\phi)$.
- *Explicit substitution* $P[a \mapsto T]$ represents $\phi[\![a \mapsto t]\!]$.

Introduction (2)

The meta-level assertions in first-order logic

- $\phi \supset \psi \supset \phi$
- if $a \notin fn(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin fn(\phi)$ then $\phi \supset \phi[\![a \mapsto t]\!]$
- if $b \notin fn(\phi)$ then $\forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]$

correspond to valid assertions in one-and-a-halfth-order logic:

- $P \supset Q \supset P$
- $a\#P \rightarrow P \supset \forall[a]P$
- $a\#P \rightarrow P \supset P[a \mapsto T]$
- $b\#P \rightarrow \forall[a]P \supset \forall[b]P[a \mapsto b]$

Introduction (3)

In sequent derivations of one-and-a-halfth-order logic:

- *Contexts of freshnesses* are added to the sequents.
- *Derivability of freshnesses* are added as side-conditions.
- *Substitutional equivalence on terms* is added as two derivation rules, taking care of α-equivalence and substitution.

Introduction (4)

The (schematic) derivations in first-order logic

$$\dfrac{\dfrac{\dfrac{}{\psi, \phi \vdash \phi}\,(\textbf{Ax})}{\phi \vdash \psi \supset \phi}\,(\supset\textbf{R})}{\vdash \phi \supset \psi \supset \phi}\,(\supset\textbf{R}) \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{p}(d), \mathsf{p}(c) \vdash \mathsf{p}(c)}\,(\textbf{Ax})}{\mathsf{p}(c) \vdash \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\textbf{R})}{\vdash \mathsf{p}(c) \supset \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\textbf{R})$$

correspond to valid derivations in one-and-a-halfth-order logic:

$$\dfrac{\dfrac{\dfrac{}{Q, P \vdash_{\emptyset} P}\,(\textbf{Ax})}{P \vdash_{\emptyset} Q \supset P}\,(\supset\textbf{R})}{\vdash_{\emptyset} P \supset Q \supset P}\,(\supset\textbf{R}) \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{p}(d), \mathsf{p}(c) \vdash_{\emptyset} \mathsf{p}(c)}\,(\textbf{Ax})}{\mathsf{p}(c) \vdash_{\emptyset} \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\textbf{R})}{\vdash_{\emptyset} \mathsf{p}(c) \supset \mathsf{p}(d) \supset \mathsf{p}(c)}\,(\supset\textbf{R})$$

Introduction (5)

The (schematic) derivations in first-order logic, where $b \notin fn(\phi)$,

$$\dfrac{\dfrac{}{\forall a.\phi \vdash \forall b.\phi[\![a \mapsto b]\!]}\,(\mathbf{Ax})}{\vdash \forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{}{\forall c.\mathrm{p}(c) \vdash \forall d.\mathrm{p}(d)}\,(\mathbf{Ax})}{\vdash \forall c.\mathrm{p}(c) \supset \forall d.\mathrm{p}(d)}\,(\supset\mathbf{R})$$

correspond to valid derivations in one-and-a-halfth-order logic:

$$\dfrac{\dfrac{\dfrac{}{\forall[a]P \vdash_{b\#P} \forall[a]P}\,(\mathbf{Ax})}{\forall[a]P \vdash_{b\#P} \forall[b]P[a \mapsto b]}\,(\mathbf{StructR})\ (1)}{\vdash_{b\#P} \forall[a]P \supset \forall[b]P[a \mapsto b]}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{\dfrac{}{\forall[c]\mathrm{p}(c) \vdash_{\emptyset} \forall[c]\mathrm{p}(c)}\,(\mathbf{Ax})}{\forall[c]\mathrm{p}(c) \vdash_{\emptyset} \forall[d]\mathrm{p}(d)}\,(\mathbf{StructR})\ (2)}{\vdash_{\emptyset} \forall[c]\mathrm{p}(c) \supset \forall[d]\mathrm{p}(d)}\,(\supset\mathbf{R})$$

(1) $b\#P \vdash_{\mathbf{SUB}} \forall[a]P = \forall[b]P[a \mapsto b]$

(2) $\emptyset \vdash_{\mathbf{SUB}} \forall[c]\mathrm{p}(c) = \forall[d]\mathrm{p}(d)$

Formal syntax
Nominal terms

We use **Nominal Terms** to specify the syntax, since they have built-in support for:

- meta-variables
- binding
- freshness

Nominal terms allow for a direct and natural representation of systems with binding.

Nominal terms are first-order, not higher-order.

Formal syntax

Sorts, atoms and unknowns

Base sorts $\mathbb{F}$ for 'formulas' and $\mathbb{T}$ for 'terms'.

Atomic sort $\mathbb{A}$ for the object-level variables.

Sorts τ:

$$\tau ::= \mathbb{F} \mid \mathbb{T} \mid \mathbb{A} \mid [\mathbb{A}]\tau$$

Atoms $a, b, c, \ldots$ have sort $\mathbb{A}$.
They represent *object-level* variable symbols.

Unknowns $X, Y, Z, \ldots$ have sort τ.
They represent *meta-level* variable symbols.
Let P, Q, R be unknowns of sort $\mathbb{F}$, and T, U of sort $\mathbb{T}$.

Formal syntax
Terms

We call $\pi \cdot X$ a **moderated unknown**.
This represents the **permutation of atoms** π acting on an unknown term. Write X when π is the *identity*.

Term-formers are of the form $\mathsf{f}_{(\tau_1,\ldots,\tau_n)\tau}$.

Terms t, subscripts indicate sorting rules:

$$t \ ::= a_{\mathbb{A}} \mid (\pi \cdot X_\tau)_\tau \mid ([a_{\mathbb{A}}]t_\tau)_{[\mathbb{A}]\tau} \mid (\mathsf{f}_{(\tau_1,\ldots,\tau_n)\tau}(t^1_{\tau_1}, \ldots, t^n_{\tau_n}))_\tau$$

We often drop the sorting subscripts:

$$t \ ::= a \mid \pi \cdot X \mid [a]t \mid \mathsf{f}(t_1, \ldots, t_n)$$

Write f for $\mathsf{f}()$ if $n = 0$.

Formal syntax

Terms (2)

Term-formers for one-and-a-halfth-order logic:

- $\bot_{()\mathbb{F}}$: *false*
- $\supset_{(\mathbb{F},\mathbb{F})\mathbb{F}}$: *implication*, write $\supset(\phi,\psi)$ as $\phi \supset \psi$
- $\forall_{([\mathbb{A}]\mathbb{F})\mathbb{F}}$: *universal quantification*, write $\forall([a]\phi)$ as $\forall[a]\phi$
- $\approx_{(\mathbb{T},\mathbb{T})\mathbb{F}}$: *object-level equality*, write $\approx(t,u)$ as $t \approx u$
- $\mathsf{var}_{(\mathbb{A})\mathbb{T}}$: *variable casting*, write $\mathsf{var}(a)$ as a
- $\mathsf{sub}_{([\mathbb{A}]\tau,\mathbb{T})\tau}$, where $\tau \in \{\mathbb{T},[\mathbb{A}]\mathbb{T},\mathbb{F},[\mathbb{A}]\mathbb{F}\}$:
  explicit substitution, write $\mathsf{sub}([a]v,t)$ as $v[a \mapsto t]$
- $\mathsf{p}_{1\,(\mathbb{T},\ldots,\mathbb{T})\mathbb{F}},\ldots,\mathsf{p}_{n\,(\mathbb{T},\ldots,\mathbb{T})\mathbb{F}}$: *object-level predicate term-formers*
- $\mathsf{f}_{1\,(\mathbb{T},\ldots,\mathbb{T})\mathbb{T}},\ldots,\mathsf{f}_{m\,(\mathbb{T},\ldots,\mathbb{T})\mathbb{T}}$: *object-level term-formers*

Formal syntax
Terms (3)

Sugar:

$$\top \text{ is } \bot \supset \bot \qquad \neg\phi \text{ is } \phi \supset \bot$$

$$\phi \wedge \psi \text{ is } \neg(\phi \supset \neg\psi) \qquad \phi \vee \psi \text{ is } \neg\phi \supset \psi$$

$$\phi \Leftrightarrow \psi \text{ is } (\phi \supset \psi) \wedge (\psi \supset \phi) \qquad \exists[a]\phi \text{ is } \neg\forall[a]\neg\phi$$

Descending order of operator precedence:

$$[a]\_\,,\ \ \_[\_ \mapsto \_],\ \approx,\ \{\neg, \forall, \exists\},\ \{\wedge, \vee\},\ \supset,\ \Leftrightarrow$$

$\wedge$, $\vee$, $\supset$ and $\Leftrightarrow$ associate to the right.

Formal syntax
Terms (3)

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$\wedge$, $\vee$, $\supset$ and $\Leftrightarrow$ associate to the right.

We may call terms of sort $\mathbb{F}$ **formulas**.
Example formulas:

$$P \supset Q \supset P \qquad P \supset \forall[a]P \qquad P \supset P[a \mapsto T] \qquad \forall[a]P \supset \forall[b]P[a \mapsto b]$$

Formal syntax
Freshness and terms-in-context

Freshness (assertions) $a\#t$, which means 'a is fresh for t.
If t is an unknown X, the freshness is called **primitive**.

A **freshness context** Δ is a set of *primitive* freshnesses.

Example freshness contexts:

$$\emptyset \qquad a\#X \qquad a\#P, b\#Q$$

We call $\Delta \rightarrow t$ a **term-in-context**; write t if $\Delta = \emptyset$.

Formal syntax
Assertions

Terms-in-context of sort $\mathbb{F}$ represent meta-level assertions of first-order logic. For example:

- $P \supset Q \supset P$
- $a \# P \rightarrow P \supset \forall[a]P$
- $a \# P \rightarrow P \supset P[a \mapsto T]$
- $b \# P \rightarrow \forall[a]P \supset \forall[b]P[a \mapsto b]$

Formal syntax
Assertions

Terms-in-context of sort $\mathbb{F}$ represent meta-level assertions of first-order logic. For example:

- $P \supset Q \supset P$
- $a\#P \to P \supset \forall[a]P$
- $a\#P \to P \supset P[a \mapsto T]$
- $b\#P \to \forall[a]P \supset \forall[b]P[a \mapsto b]$

represent

- $\phi \supset \psi \supset \phi$
- if $a \notin fn(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin fn(\phi)$ then $\phi \supset \phi[\![a \mapsto t]\!]$
- if $b \notin fn(\phi)$ then $\forall a.\phi \supset \forall b.\phi[\![a \mapsto b]\!]$

Derivability

Sequents

Let **(formula) contexts** Φ, Ψ be finite sets of formulas.
For example:

$$\emptyset \qquad \phi \qquad \phi, \Phi \qquad \Phi, \Phi'$$

A **sequent** is a triple $\Phi \vdash_{\vartriangle} \Psi$.
We may omit empty formula contexts, e.g. writing $\vdash_{\vartriangle}$ for $\emptyset \vdash_{\vartriangle} \emptyset$.

Derivability
Sequent calculus

Rules resembling Gentzen's sequent calculus for first-order logic:

$$\frac{}{\phi,\ \Phi \vdash_{\Delta} \Psi,\ \phi}\ (\mathbf{Ax}) \qquad \frac{}{\bot,\ \Phi \vdash_{\Delta} \Psi}\ (\bot\mathbf{L})$$

$$\frac{\Phi \vdash_{\Delta} \Psi,\ \phi \quad \psi,\ \Phi \vdash_{\Delta} \Psi}{\phi \supset \psi,\ \Phi \vdash_{\Delta} \Psi}\ (\supset\mathbf{L}) \qquad \frac{\phi,\ \Phi \vdash_{\Delta} \Psi,\ \psi}{\Phi \vdash_{\Delta} \Psi,\ \phi \supset \psi}\ (\supset\mathbf{R})$$

$$\frac{\phi[a \mapsto t],\ \Phi \vdash_{\Delta} \Psi}{\forall[a]\phi,\ \Phi \vdash_{\Delta} \Psi}\ (\forall\mathbf{L}) \qquad \frac{\Phi \vdash_{\Delta} \Psi,\ \psi}{\Phi \vdash_{\Delta} \Psi,\ \forall[a]\psi}\ (\forall\mathbf{R}) \quad (\Delta \vdash a\#\Phi, \Psi)$$

$$\frac{\phi[a \mapsto t'],\ \Phi \vdash_{\Delta} \Psi}{t' \approx t,\ \phi[a \mapsto t],\ \Phi \vdash_{\Delta} \Psi}\ (\approx\mathbf{L}) \qquad \frac{}{\Phi \vdash_{\Delta} \Psi,\ t \approx t}\ (\approx\mathbf{R})$$

Derivability

Sequent calculus (2)

Other rules:

$$\frac{\phi', \Phi \vdash_{\Delta} \Psi}{\phi, \Phi \vdash_{\Delta} \Psi}\ (\textbf{StructL}) \quad (\Delta \vdash_{\textsf{SUB}} \phi' = \phi)$$

$$\frac{\Phi \vdash_{\Delta} \Psi, \psi'}{\Phi \vdash_{\Delta} \Psi, \psi}\ (\textbf{StructR}) \quad (\Delta \vdash_{\textsf{SUB}} \psi' = \psi)$$

$$\frac{\Phi \vdash_{\Delta \cup \{a\#X_1, \ldots, a\#X_n\}} \Psi}{\Phi \vdash_{\Delta} \Psi}\ (\textbf{Fresh}) \quad (a \notin \Phi, \Psi, \Delta)$$

$$\frac{\Phi \vdash_{\Delta} \Psi, \phi \qquad \phi', \Phi \vdash_{\Delta} \Psi}{\Phi \vdash_{\Delta} \Psi}\ (\textbf{Cut}) \quad (\Delta \vdash_{\textsf{SUB}} \phi = \phi')$$

Derivability

Example derivations in the sequent calculus

Sequent derivation of $a\#P \to P \supset \forall[a]P$:

$$\dfrac{\dfrac{\dfrac{}{P \vdash_{a\#P} P}\,(\mathbf{Ax})}{P \vdash_{a\#P} \forall[a]P}\,(\forall\mathbf{R}) \quad (a\#P \vdash a\#P)}{\vdash_{a\#P} P \supset \forall[a]P}\,(\supset\mathbf{R})$$

Derivation of $a\#P \to P \supset P[a \mapsto T]$:

$$\dfrac{\dfrac{\dfrac{}{P \vdash_{a\#P} P}\,(\mathbf{Ax})}{P \vdash_{a\#P} P[a \mapsto T]}\,(\mathbf{StructR}) \quad (a\#P \vdash_{\mathsf{SUB}} P = P[a \mapsto T])}{\vdash_{a\#P} P \supset P[a \mapsto T]}\,(\supset\mathbf{R})$$

Derivability

Freshness

Write $\Delta \vdash a\#t$ when $a\#t$ **is derivable** from Δ using the following inference rules:

$$\frac{}{a\#b}\,(\#\mathbf{ab}) \qquad \frac{\pi^{-1}(a)\#X}{a\#\pi\cdot X}\,(\#\mathbf{X})$$

$$\frac{}{a\#[a]t}\,(\#[]\mathbf{a}) \qquad \frac{a\#t}{a\#[b]t}\,(\#[]\mathbf{b}) \qquad \frac{a\#t_1 \ \cdots \ a\#t_n}{a\#\mathsf{f}(t_1,\ldots,t_n)}\,(\#\mathbf{f})$$

Here, a and b range over *distinct* atoms.

Derivability
Freshness

Write $\Delta \vdash a\#t$ when $a\#t$ **is derivable** from Δ using the following inference rules:

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$$\frac{}{a\#[a]t}\,(\#[]\mathbf{a}) \qquad \frac{a\#t}{a\#[b]t}\,(\#[]\mathbf{b}) \qquad \frac{a\#t_1 \ \cdots \ a\#t_n}{a\#\mathsf{f}(t_1,\ldots,t_n)}\,(\#\mathbf{f})$$

Here, a and b range over *distinct* atoms.

Examples:

$$\vdash a\#b \qquad \vdash a\#\forall[a]P \qquad a\#P \vdash a\#\forall[b]P$$

Derivability

Equality

Equality (assertions) $t = u$, where t and u are of the same sort. Write $\Delta \vdash_{\mathrm{SUB}} t = u$ when $t = u$ **is derivable from** Δ using the following inference rules, where A are **axioms** from SUB only:

$$\frac{}{t = t}\ \textbf{(refl)} \qquad \frac{t = u}{u = t}\ \textbf{(symm)} \qquad \frac{t = u \quad u = v}{t = v}\ \textbf{(tran)}$$

$$\frac{t = u}{C[t] = C[u]}\ \textbf{(cong)} \qquad \frac{a\#t \quad b\#t}{(a\ b)\cdot t = t}\ \textbf{(perm)}$$

$$\frac{\Delta^{\pi}\sigma}{t^{\pi}\sigma = u^{\pi}\sigma}\ (\mathbf{ax_A})\ A \text{ is } \Delta \to t = u$$

$$\frac{\begin{array}{cc}[a\#X_1, \ldots, a\#X_n] & \Delta \\ \vdots & \\ t = u & \end{array}}{t = u}\ \textbf{(fr)} \quad (a \notin t, u, \Delta)$$

Derivability

Equality (2)

Axioms of **theory** SUB:

$$
\begin{array}{rrcl}
(\textbf{var} \mapsto) & a[a \mapsto T] & = & T \\
(\# \mapsto) & a\#X \to X[a \mapsto T] & = & X \\
(\textbf{f} \mapsto) & \mathsf{f}(X_1, \ldots, X_n)[a \mapsto T] & = & \mathsf{f}(X_1[a \mapsto T], \ldots, X_n[a \mapsto T]) \\
(\textbf{abs} \mapsto) & b\#T \to ([b]X)[a \mapsto T] & = & [b](X[a \mapsto T]) \\
(\textbf{ren} \mapsto) & b\#X \to X[a \mapsto b] & = & (b\ a) \cdot X
\end{array}
$$

Derivability

Equality (2)

Axioms of **theory** SUB:

$$
\begin{array}{rrcl}
(\mathbf{var} \mapsto) & a[a \mapsto T] & = & T \\
(\# \mapsto) & a\#X \to X[a \mapsto T] & = & X \\
(\mathbf{f} \mapsto) & \mathsf{f}(X_1, \ldots, X_n)[a \mapsto T] & = & \mathsf{f}(X_1[a \mapsto T], \ldots, X_n[a \mapsto T]) \\
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\end{array}
$$

Examples:

$$b\#P \vdash_{\mathsf{SUB}} \forall[a]P = \forall[b]P[a \mapsto b]$$

$$\vdash_{\mathsf{SUB}} X[a \mapsto a] = X$$

$$a\#Y \vdash_{\mathsf{SUB}} Z[a \mapsto X][b \mapsto Y] = Z[b \mapsto Y][a \mapsto X[b \mapsto Y]]$$

Derivability

Equality (2)

Axioms of **theory** SUB:

$$
\begin{array}{rrcl}
(\mathbf{var} \mapsto) & a[a \mapsto T] & = & T \\
(\# \mapsto) & a\#X \to X[a \mapsto T] & = & X \\
(\mathbf{f} \mapsto) & \mathsf{f}(X_1, \ldots, X_n)[a \mapsto T] & = & \mathsf{f}(X_1[a \mapsto T], \ldots, X_n[a \mapsto T]) \\
(\mathbf{abs} \mapsto) & b\#T \to ([b]X)[a \mapsto T] & = & [b](X[a \mapsto T]) \\
(\mathbf{ren} \mapsto) & b\#X \to X[a \mapsto b] & = & (b\ a) \cdot X
\end{array}
$$

Examples:

$$b\#P \vdash_{\mathsf{SUB}} \forall[a]P = \forall[b]P[a \mapsto b]$$

$$\vdash_{\mathsf{SUB}} X[a \mapsto a] = X$$

$$a\#Y \vdash_{\mathsf{SUB}} Z[a \mapsto X][b \mapsto Y] = Z[b \mapsto Y][a \mapsto X[b \mapsto Y]]$$

Nominal Algebra is the theory of equality on nominal terms.

Proof-theoretical properties

Permutation and instantiation

We may permute atoms and instantiate unknowns in derivations.

Theorem

If Π *is a valid derivation of* $\Phi \vdash_{\Delta} \Psi$*,*
then Π^{π} *is a valid derivation of* $\Phi^{\pi} \vdash_{\Delta^{\pi}} \Psi^{\pi}$*.*

Theorem

If Π *is a valid derivation of* $\Phi \vdash_{\Delta} \Psi$ *and* $\Delta' \vdash \Delta\sigma$*,*
then $\Pi(\sigma, \Delta')$ *is a valid derivation of* $\Phi\sigma \vdash_{\Delta'} \Psi\sigma$*.*

$\Pi(\sigma, \Delta')$ is Π in which:

- each unknown X is replaced by $\sigma(X)$
- each freshness context Δ is replaced by Δ'

Proof-theoretical properties
Instantiation example

Take the following derivations:

$$\dfrac{\dfrac{\dfrac{}{P \vdash_{a\#P} P}\,(\mathbf{Ax})}{P \vdash_{a\#P} P[a \mapsto T]}\,(\mathbf{StructR})\ (1)}{\vdash_{a\#P} P \supset P[a \mapsto T]}\,(\supset\mathbf{R}) \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{p}(c) \vdash_{\emptyset} \mathsf{p}(c)}\,(\mathbf{Ax})}{\mathsf{p}(c) \vdash_{\emptyset} \mathsf{p}(c)[a \mapsto d]}\,(\mathbf{StructR})\ (2)}{\vdash_{\emptyset} \mathsf{p}(c) \supset \mathsf{p}(c)[a \mapsto d]}\,(\supset\mathbf{R})$$

(1) $a\#P \vdash_{\mathbf{SUB}} P = P[a \mapsto T])$

(2) $\emptyset \vdash_{\mathbf{SUB}} \mathsf{p}(c) = \mathsf{p}(c)[a \mapsto d])$

The derivation on the right is an instance of the one on the left:

- call the left derivation Π
- then the right one is $\Pi([\mathsf{p}(c)/P, d/T], \emptyset)$, which is valid because $\emptyset \vdash a\#P[\mathsf{p}(c)/P, d/T]$, i.e. $\emptyset \vdash a\#\mathsf{p}(c)$

Proof-theoretical properties

Cut elimination

Theorem (Cut elimination)

The (**Cut**) *rule is admissible in the system without it.*

Proof-theoretical properties
Cut elimination

Theorem (Cut elimination)

The (**Cut**) *rule is admissible in the system without it.*

Corollary

The sequent calculus is consistent, i.e. $\vdash_{\Delta}$ *can never be derived.*

Axiomatisation

Theory FOL

Theory FOL extends theory SUB with the following axioms:

$$P \supset Q \supset P \;=\; \top \quad \neg\neg P \supset P \;=\; \top \quad \top \supset P \;=\; P \qquad \textbf{(Props)}$$

$$(P \supset Q) \supset (Q \supset R) \supset (P \supset R) \;=\; \top \quad \bot \supset P \;=\; \top$$

$$\forall[a]P \supset P[a \mapsto T] \;=\; \top \qquad \textbf{(Quants)}$$

$$\forall[a](P \wedge Q) \Leftrightarrow \forall[a]P \wedge \forall[a]Q \;=\; \top$$

$$a\#P \;\rightarrow\; \forall[a](P \supset Q) \Leftrightarrow P \supset \forall[a]Q \;=\; \top$$

$$T \approx T = \top \quad U \approx T \wedge P[a \mapsto T] \supset P[a \mapsto U] = \top \qquad \textbf{(Eq)}$$

Axioms of the form $\phi = \top$ intuitively mean 'ϕ is true'.
Note that this is a finite number of axioms.

Axiomatisation
Equivalence with sequent calculus

Sequent and equational derivability are equivalent:

Theorem
For all formula contexts Φ, Ψ and freshness contexts Δ:

$$\Phi \vdash_{\Delta} \Psi \textit{ is derivable} \quad \textit{iff} \quad \Delta \vdash_{\mathsf{FOL}} \Phi^{\wedge} \supset \Psi^{\vee} = \top .$$

Here:

- $\Phi^{\wedge}$ is the *conjunction* of all formulas in Φ
- $\Psi^{\vee}$ the *disjunction* of all formulas in Ψ

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Corollary

Theory FOL *is consistent, i.e.* $\Delta \vdash_{\mathsf{FOL}} \top = \bot$ *does not hold.*

Relation to First-order Logic

Call a term or a formula context **ground** if it does not contain *unknowns* or *explicit substitutions*.
Call $\Phi \vdash \Psi$ a **first-order sequent** when Φ and Ψ are ground.
Gentzen's sequent calculus for first-order logic:

$$\frac{}{\phi,\ \Phi \vdash \Psi,\ \phi}\ (\mathbf{Ax}) \qquad \frac{}{\bot,\ \Phi \vdash \Psi}\ (\bot\mathbf{L})$$

$$\frac{\Phi \vdash \Psi,\ \phi \qquad \psi,\ \Phi \vdash \Psi}{\phi \supset \psi,\ \Phi \vdash \Psi}\ (\supset\mathbf{L}) \qquad \frac{\phi,\ \Phi \vdash \Psi,\ \psi}{\Phi \vdash \Psi,\ \phi \supset \psi}\ (\supset\mathbf{R})$$

$$\frac{\phi[\![a \mapsto t]\!],\ \Phi \vdash \Psi}{\forall a.\phi,\ \Phi \vdash \Psi}\ (\forall\mathbf{L}) \qquad \frac{\Phi \vdash \Psi,\ \phi}{\Phi \vdash \Psi,\ \forall a.\phi}\ (\forall\mathbf{R}) \quad (a \notin fn(\Phi, \Psi))$$

$$\frac{\phi[\![a \mapsto t']\!],\ \Phi \vdash \Psi}{t' \approx t,\ \phi[\![a \mapsto t]\!],\ \Phi \vdash \Psi}\ (\approx \mathbf{L}) \qquad \frac{}{\Phi \vdash \Psi,\ t \approx t}\ (\approx \mathbf{R})$$

Relation to First-order Logic (2)

Note that:

- we write $\forall a.\phi$ for $\forall[a]\phi$
- $[\![a \mapsto t]\!]$ is capture-avoiding substitution
- $a \notin fn(\phi)$ is 'a does not occur in the free names of ϕ'
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On ground terms, one-and-a-halfth-order logic is first-order logic:

Theorem

$\Phi \vdash \Psi$ is derivable in the sequent calculus for first-order logic, iff $\Phi \vdash_{\emptyset} \Psi$ is derivable in the sequent calculus for one-and-a-halfth-order logic.

Semantics

For *closed* terms t, its **ground form** $t[\![]\!]$ is t in which each explicit substitution $v[a \mapsto u]$ is replaced by $v[\![a \mapsto u]\!]$.

Lemma

For closed terms t, $\vdash_{\mathrm{SUB}} t = t[\![]\!]$.

A term-in-context $\Delta \longrightarrow \phi$ is **valid** iff $\phi\sigma[\![]\!]$ is valid in first-order logic for all instantiations σ such that $\phi\sigma$ is closed and $\vdash \Delta\sigma$ holds.

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The sequent calculus for one-and-a-halfth-order logic is sound for this semantics:

Theorem

If $\vdash_{\Delta} \phi$ *is derivable then* $\Delta \to \phi$ *is valid.*

Conclusions

Using nominal terms, we can:

- *accurately* represent systems with binding:
  e.g. explicit substitution and first-order logic
- specify *novel* systems with their own mathematical interest:
  e.g. one-and-a-halfth-order logic

One-and-a-halfth-order logic:

- makes meta-level concepts of first-order logic *explicit*
- has a sequent calculus with *syntax-directed* rules
- has a *semantics* in first-order logic
- has a *finite* equational axiomatisation
- is the *result* of axiomatising first-order logic in nominal algebra

Related work

In **Second-Order logic (SOL)** we can quantify over predicates *anywhere*: more expressive than one-and-a-halfh-order logic.

On the other hand, we can easily extend theory FOL with *one* axiom to express the principle of induction on natural numbers:

$$P[a \mapsto 0] \wedge \forall[a](P \supset P[a \mapsto succ(a)]) \supset \forall[a]P \ = \ \top.$$

Higher-Order Logic (HOL) is type raising, while our logic is *not*:

- $P[a \mapsto t]$ corresponds to $f(t)$ in HOL, where $f : \mathbb{T} \rightarrow \mathbb{F}$
- $P[a \mapsto t][a' \mapsto t']$ corresponds to $f'(t)(t')$ where $f' : \mathbb{T} \rightarrow \mathbb{T} \rightarrow \mathbb{F}$

One-and-a-halfth-order logic is not a subset of SOL or HOL because of freshnesses.

Future work

Topics:

- Completeness of the sequent calculus with respect to the semantics.
- Let unknowns range over *sequent derivations*, and establish a Curry-Howard correspondence (term-in-contexts as types, derivations as terms).
- Two-and-a-halfth-order logic (where you can abstract X)?
- Implementation and automation?

Further reading

- Murdoch J. Gabbay, Aad Mathijssen:
  One-and-a-halfth-order Logic.
  PPDP'06.
- Murdoch J. Gabbay, Aad Mathijssen:
  Capture-Avoiding Substitution as a Nominal Algebra.
  ICTAC'06.
- Murdoch J. Gabbay, Aad Mathijssen:
  Nominal Algebra.
  Submitted STACS'07.

Just to scare you

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P[b \mapsto c][a \mapsto c] \vdash_{c\#P} P[b \mapsto c][a \mapsto c]}\,(\mathbf{Ax})}{\forall[a]P[b \mapsto c] \vdash_{c\#P} P[b \mapsto c][a \mapsto c]}\,(\forall\mathbf{L})}{(\forall[a]P)[b \mapsto c] \vdash_{c\#P} P[b \mapsto a][a \mapsto c]}\,(\mathbf{StructL})\quad(1)}{\forall[b]\forall[a]P \vdash_{c\#P} P[b \mapsto c][a \mapsto c]}\,(\forall\mathbf{L})}{\forall[b]\forall[a]P \vdash_{c\#P} \forall[c]P[b \mapsto c][a \mapsto c]}\,(\forall\mathbf{R})\quad(2)}{\forall[b]\forall[a]P \vdash_{c\#P} \forall[a]P[b \mapsto a]}\,(\mathbf{StructR})\quad(3)}{\forall[b]\forall[a]P \vdash_{\emptyset} \forall[a]P[b \mapsto a]}\,(\mathbf{Fresh})\quad(4)
$$

Side-conditions:

(1) $c\#P \vdash_{\mathbf{SUB}} \forall[a]P[b \mapsto c] = (\forall[a]P)[b \mapsto c]$

(2) $c\#P \vdash c\#\forall[b]\forall[a]P$

(3) $c\#P \vdash_{\mathbf{SUB}} \forall[c]P[b \mapsto c][a \mapsto c] = \forall[a]P[b \mapsto a]$

(4) $c \notin \forall[b]\forall[a]P,\ \forall[a]P[b \mapsto a]$