

# LOGICAL CALCULI FOR REASONING WITH BINDING

ADRIANUS HUBERTUS JOHANNES MATHIJSEN

# Logic

$$\frac{}{b[a \mapsto a]} \quad (Ax)$$

$$\frac{a \neq b}{b[a \mapsto a] = b} \quad (Ax)$$

$$\frac{}{(\lambda[a])b = b}$$

$$\frac{}{((\lambda[a])b)a = ba}$$

$$\frac{}{(\lambda((\lambda[a])b))a = (a)(b \ a)}$$

$$\frac{}{(\lambda((\lambda[a])b))a = \lambda[a](b \ a)}$$

$$\lambda[a](\lambda[a](b))a = b$$

$$\frac{[b \neq X]}{b \neq [a]X} \quad (\neq I)$$

$$\frac{a \cdot X = [a] \cdot X}{\neq [b](b \ a) \cdot X} \quad (\text{SYMP})$$

$$\frac{}{\neq (b \ a) \cdot X [b \mapsto a]} \quad (\text{CONJ})$$

$$\frac{X [a \mapsto a] = X}{X [a \mapsto a] = X} \quad (\text{VR})$$

$$\frac{X [a \mapsto a] = X}{X [a \mapsto a] = X} \quad (\text{VR})$$

$$\frac{}{P \mapsto \neq P Q} \quad (Ax)$$

$$\frac{}{Q, P \mapsto \neq P Q} \quad (Ax)$$

$$\frac{P, P \neq Q \mapsto \neq P Q}{P \neq Q \mapsto \neq P Q} \quad (\text{STRUC})$$

$$\frac{P \neq Q \mapsto \neq P Q}{P \neq Q \mapsto \neq P Q} \quad (\text{STRUC})$$

$$\frac{P \neq Q \mapsto \neq P Q}{P \neq Q \mapsto \neq P Q} \quad (\text{STRUC})$$

$$\frac{P \neq Q \mapsto \neq P Q}{P \neq Q \mapsto \neq P Q} \quad (\text{STRUC})$$

$$\frac{}{a \neq a}$$

$$\lambda[a](a) = a$$

$$\frac{P, Q \mapsto \neq P Q}{P, Q \mapsto \neq P Q} \quad (Ax)$$

$$\frac{P, Q \mapsto \neq P Q}{P, Q \mapsto \neq P Q} \quad (\text{VR})$$

$$\frac{P, Q \mapsto \neq P Q}{P, Q \mapsto \neq P Q} \quad (\text{STRUC})$$

$$\frac{P, Q \mapsto \neq P Q}{P, Q \mapsto \neq P Q} \quad (\text{FR})$$

$$\frac{a \neq a \mapsto \neq a X}{(a \neq a) \cdot X [b \mapsto a] = X} \quad (\text{AXREN})$$

$$\frac{(a \neq a) \cdot X [b \mapsto a] = X}{(a \neq a) \cdot X [b \mapsto a] = X} \quad (\text{TRAI})$$

$$\frac{}{X [a \mapsto a] = X} \quad (\text{TRAI})$$

Handwritten notes and scribbles at the bottom right of the page.

# Logic

Logic studies reasoning.

$$\frac{a \neq b}{\lambda[a]b = b}$$
$$\frac{a \neq b}{\lambda[a]b = b}$$
$$\lambda[a]((\lambda[a]b) \cap d) = (\lambda[a]b) \cap d$$
$$\lambda[a]((\lambda[a]b) \cap d) = \lambda[a](b \cap d)$$
$$\lambda[a](\lambda[a]((\lambda[a]b) \cap d)) = b$$

$$\frac{b \neq X}{b \neq [a]X} \quad (\neq [I])$$
$$\frac{a \cdot X = [a] \cdot X}{a \neq [b](b \cdot a) \cdot X} \quad (\text{SYMP})$$
$$\frac{a \neq [b](b \cdot a) \cdot X}{a \neq (b \cdot a) \cdot X [b \rightarrow a]} \quad (\text{CONJEP})$$
$$\frac{X [a \rightarrow b] = X}{X [b \rightarrow a] = X} \quad (\text{TRAI})$$
$$\frac{X [a \rightarrow b] = X}{X [b \rightarrow a] = X} \quad (\text{VR})$$

$$\frac{}{P \rightarrow_{\neq} P} \quad (\text{Ax})$$
$$\frac{Q, P \rightarrow_{\neq} P}{Q} \quad (\text{Ax})$$
$$\frac{P, P \rightarrow_{\neq} Q}{P \rightarrow_{\neq} Q} \quad (\text{DL})$$
$$\frac{P \rightarrow_{\neq} Q, Q \rightarrow_{\neq} P}{P \rightarrow_{\neq} P} \quad (\text{STRUC})$$
$$\frac{P \rightarrow_{\neq} Q, Q \rightarrow_{\neq} P}{P \rightarrow_{\neq} P} \quad (\text{DL})$$
$$\frac{P \rightarrow_{\neq} Q, Q \rightarrow_{\neq} P}{P \rightarrow_{\neq} P} \quad (\text{VR})$$
$$\lambda[a]b = b$$
$$\frac{P, Q \rightarrow_{\neq} P}{P, Q \rightarrow_{\neq} P} \quad (\text{Ax})$$
$$\frac{P, Q \rightarrow_{\neq} P}{P, Q \rightarrow_{\neq} P} \quad (\text{VR})$$
$$\frac{P, Q \rightarrow_{\neq} P}{P, Q \rightarrow_{\neq} P} \quad (\text{STRUC})$$
$$\frac{P, Q \rightarrow_{\neq} P}{P, Q \rightarrow_{\neq} P} \quad (\text{FR})$$
$$\frac{a \neq [b]X}{(a \cdot b) \cdot X [b \rightarrow a] = X} \quad (\text{AXREN})$$
$$\frac{a \neq [b]X}{(a \cdot b) \cdot X [b \rightarrow a] = X} \quad (\text{TRAI})$$

# Logic

Logic studies **reasoning**.

## Example

"If the committee is satisfied then I will obtain my degree."

# Logic

Logic studies **reasoning**.

## Example

“If the committee is satisfied then I will obtain my degree.”

Expressed in logic by the following formula:

$\text{committee\_satisfied} \Rightarrow \text{obtain\_degree}$

# Logic

Logic studies **reasoning**.

## Example

“If the committee is satisfied then I will obtain my degree.”

Expressed in logic by the following formula:

$\text{committee\_satisfied} \Rightarrow \text{obtain\_degree}$

“If all human beings are mortal then Socrates is mortal.”

# Logic

Logic studies **reasoning**.

## Example

“If the committee is satisfied then I will obtain my degree.”

Expressed in logic by the following formula:

$\text{committee\_satisfied} \Rightarrow \text{obtain\_degree}$

“If all human beings are mortal then Socrates is mortal.”

Expressed in logic by the following formula:

$\forall x \in \text{Humans} \text{ mortal}(x) \Rightarrow \text{mortal}(\text{Socrates})$

# Reasoning about logics

$$\frac{b \mid (a \rightarrow a)}{b \mid (a \rightarrow a)} \text{ (ID)}$$

$$\frac{(\lambda a) \mid a = b}{(\lambda a) \mid a = b} \text{ (TRAI)}$$

$$\frac{(\lambda a) \mid a = b}{(\lambda a) \mid a = b} \text{ (TRAI)}$$

$$\frac{(\lambda a) \mid a = b}{(\lambda a) \mid a = b} \text{ (TRAI)}$$

$$\frac{(\lambda a) \mid a = b}{(\lambda a) \mid a = b} \text{ (TRAI)}$$

$$\frac{(\lambda a) \mid a = b}{(\lambda a) \mid a = b} \text{ (TRAI)}$$

$$\frac{(\lambda a) \mid a = b}{(\lambda a) \mid a = b} \text{ (TRAI)}$$

$$\frac{(\lambda a) \mid a = b}{(\lambda a) \mid a = b} \text{ (TRAI)}$$

$$\frac{P \rightarrow P \rightarrow Q}{P \rightarrow P \rightarrow Q} \text{ (Ax)}$$

$$\frac{P, P \rightarrow Q}{P \rightarrow Q} \text{ (MP)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{Q, P \rightarrow P \rightarrow Q}{Q, P \rightarrow P \rightarrow Q} \text{ (Ax)}$$

$$\frac{P, P \rightarrow Q}{P \rightarrow Q} \text{ (MP)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$

$$\frac{P \rightarrow Q, Q}{P} \text{ (MT)}$$



# Reasoning about logics

In many cases we reason about logics:

$$\phi \Rightarrow (\psi \Rightarrow \phi)$$

# Reasoning about logics

In many cases we reason about logics:

$$\phi \Rightarrow (\psi \Rightarrow \phi)$$

$\phi$  and  $\psi$  are **meta-variables** ranging over arbitrary formulas.

We have a **schema** of formulas, one for each instantiation of  $\phi$  and  $\psi$ .

## Reasoning about logics

In many cases we reason about logics:

$$\phi \Rightarrow (\psi \Rightarrow \phi)$$

$\phi$  and  $\psi$  are **meta-variables** ranging over arbitrary formulas.

We have a **schema** of formulas, one for each instantiation of  $\phi$  and  $\psi$ .

### Example

Take `committee_satisfied` for  $\phi$  and `obtain_degree` for  $\psi$ :

$$\text{committee\_satisfied} \Rightarrow (\text{obtain\_degree} \Rightarrow \text{committee\_satisfied})$$

# Reasoning about logics with binders

$$\frac{}{b[a \mapsto a] = b} \text{ (TRAI)}$$
$$\frac{(\lambda [a]) a = b}{(\lambda [a]) (b) a = b} \text{ (LAMB)}$$
$$\frac{}{(\lambda [a]) ((\lambda [a]) a) = (\lambda [a]) a} \text{ (APP)}$$
$$\frac{}{(\lambda [a]) ((\lambda [a]) a) = \lambda [a] a} \text{ (APP)}$$
$$\frac{}{\lambda [a] ((\lambda [a]) a) a = b} \text{ (APP)}$$

$$\frac{[b \neq X]}{b \neq [a] X} \text{ (NEQ)}$$
$$\frac{a \neq X}{a \neq [a] X} \text{ (SYMP)}$$
$$\frac{a \neq (b \ a) \cdot X}{a \neq (b \ a) \cdot X [b \mapsto a]} \text{ (CONJ)}$$
$$\frac{X [b \mapsto a] = X}{X [a \mapsto a] = X} \text{ (FR)}$$

$$\frac{P, Q \vdash P}{P, Q \vdash P \vee \forall a P} \text{ (VR)}$$
$$\frac{P, Q \vdash P \vee \forall a P}{P, Q \vdash P \vee \forall a P} \text{ (STR)}$$
$$\frac{P, Q \vdash P \vee \forall a P}{a \neq [a] X} \text{ (FR)}$$
$$\frac{}{(b \ a) \cdot X [b \mapsto a] = X} \text{ (AXREN)}$$
$$\frac{}{(\lambda [a]) a = a} \text{ (TRAI)}$$

# Reasoning about logics with binders

In many cases we reason about logics with **binders**, such as  $\forall$ :

$\phi \Rightarrow \forall x.\phi$  if  $x$  does not occur free in  $\phi$

$\forall x.\phi \Rightarrow \phi[t/x]$

$\phi$  is a meta-variable ranging over formulas.

$t$  is a meta-variable ranging over terms.

## Reasoning about logics with binders

In many cases we reason about logics with **binders**, such as  $\forall$ :

$\phi \Rightarrow \forall x.\phi$       if  $x$  does not occur free in  $\phi$

$\forall x.\phi \Rightarrow \phi[t/x]$

$\phi$  is a meta-variable ranging over formulas.

$t$  is a meta-variable ranging over terms.

We need to define the following concepts:

- freshness conditions: if  $x$  does not occur free in  $\phi$
- substitution  $\phi[t/x]$

# Observation

If logic teaches us to study reasoning,  
we should also study reasoning about logics.

# Logics for reasoning about logics

$\frac{}{b[a \rightarrow a]}$	$\frac{}{b[a \rightarrow a] = b}$	$\frac{}{P \vdash P} (Ax)$	$\frac{}{P \vdash P \rightarrow P} (Ax)$
$\frac{}{(\lambda[a])\delta a = b}$	$\frac{}{((\lambda[a])\delta)\delta a = b}$	$\frac{}{P \vdash Q \rightarrow (P \rightarrow Q)}$	$\frac{}{P \vdash Q \rightarrow (Q \rightarrow P)}$
$\frac{}{((\lambda[a])\delta)\delta a = b}$	$\frac{}{((\lambda[a])\delta)\delta a = \lambda[a](\delta a)}$	$\frac{}{P \vdash Q \rightarrow (Q \rightarrow P)}$	$\frac{}{P \vdash Q \rightarrow (P \rightarrow Q)}$
$\frac{}{((\lambda[a])\delta)\delta a = \lambda[a](\delta a)}$	$\frac{}{\lambda[a](\delta a) = \delta a}$	$\frac{}{P \vdash Q \rightarrow (Q \rightarrow P)}$	$\frac{}{P \vdash Q \rightarrow (P \rightarrow Q)}$
$\frac{}{\lambda[a](\lambda[a])\delta a = b}$	$\frac{}{\lambda[a](\delta a) = \delta a}$	$\frac{}{P \vdash Q \rightarrow (Q \rightarrow P)}$	$\frac{}{P \vdash Q \rightarrow (P \rightarrow Q)}$
$\frac{[b \neq X]}{b \neq [a]X} (\neq I)$	$\frac{[b \neq X]}{b \neq [a]X} (\neq O)$	$\frac{}{P, Q \vdash P \rightarrow Q} (Ax)$	$\frac{}{P, Q \vdash P \rightarrow Q} (VR)$
$\frac{a \cdot X = [a] \cdot X}{\neq [b](b \cdot a) \cdot X} (SYM)$	$\frac{a \cdot X = [a] \cdot X}{\neq [b](b \cdot a) \cdot X} (CONS)$	$\frac{}{P, Q \vdash P \rightarrow Q} (Ax)$	$\frac{}{P, Q \vdash P \rightarrow Q} (VR)$
$\frac{a \cdot X = [a] \cdot X}{\neq (b \cdot a) \cdot X [b \rightarrow a]} (CONS)$	$\frac{a \cdot X = [a] \cdot X}{\neq (b \cdot a) \cdot X [b \rightarrow a]} (CONS)$	$\frac{}{P, Q \vdash P \rightarrow Q} (Ax)$	$\frac{}{P, Q \vdash P \rightarrow Q} (VR)$
$\frac{X[a \rightarrow a] = X}{X[a \rightarrow a] = X} (FR)$	$\frac{X[a \rightarrow a] = X}{X[a \rightarrow a] = X} (FR)$	$\frac{}{P, Q \vdash P \rightarrow Q} (Ax)$	$\frac{}{P, Q \vdash P \rightarrow Q} (VR)$
$\frac{X[a \rightarrow a] = X}{X[a \rightarrow a] = X} (FR)$	$\frac{X[a \rightarrow a] = X}{X[a \rightarrow a] = X} (FR)$	$\frac{}{P, Q \vdash P \rightarrow Q} (Ax)$	$\frac{}{P, Q \vdash P \rightarrow Q} (VR)$



# Logics for reasoning about logics

Extend logics with **explicit** meta-variables  $P, Q, \dots$  and  $T, U, \dots$ :

$$P \Rightarrow (Q \Rightarrow P)$$

$$P \Rightarrow \forall x.P \quad \text{if } x \text{ does not occur free in } P$$

$$\forall x.P \Rightarrow P[T/x]$$

Side-conditions are problematic... How do we define:

- freshness conditions “if  $x$  does not occur free in  $P$ ”?
- substitution  $P[T/x]$ ?

# Logics for reasoning about logics

Extend logics with **explicit** meta-variables, freshness conditions and explicit substitutions:

$$\vdash P \Rightarrow (Q \Rightarrow P)$$

$$\vdash_{x\#P} P \Rightarrow \forall x.P$$

$$\vdash \forall x.P \Rightarrow P[x \mapsto T]$$

Problematic side-conditions are built-in:

- $\vdash_{x\#P}$  represents “if  $x$  does not occur free in  $P$ ”
- $P[x \mapsto T]$  represents  $P[T/x]$

# *Contents of the thesis*

*The thesis introduces two logics to reason about logics with binders:*

*Equational logic:*

- *equational calculus (Chapter 2)*
- *semantics in nominal sets (Chapter 3)*
- *axiomatisation of explicit substitution (Chapter 4)*

*First-order logic:*

- *sequent calculus (Chapter 5)*
- *axiomatisation of the sequent calculus (Chapter 6)*

# Conclusion

*This thesis shows how we can formalise the meta-level of logics with binding in a way that is close to informal practice.*

