LOGICAL CALCULI FOR REASONING WITH BINDING

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Expressed in logic by the following formula:

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"If all human beings are mortal then Socrates is mortal."

Expressed in logic by the following formula:

$$\forall_{x \in Humans}\, mortal(x) \Rightarrow mortal(Socrates)$$

Reasoning about logics

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Example

Take committee_satisfied for ϕ *and obtain_degree for* ψ*:*

$$\mathit{committee\_satisfied} \;\Rightarrow\; (\mathit{obtain\_degree} \;\Rightarrow\; \mathit{committee\_satisfied})$$

Reasoning about logics with binders

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In many cases we reason about logics with ***binders****, such as* $\forall$*:*

$$\phi \;\Rightarrow\; \forall x.\phi \qquad \textit{if } x \textit{ does not occur free in } \phi$$

$$\forall x.\phi \;\Rightarrow\; \phi[t/x]$$

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ϕ *is a meta-variable ranging over formulas.*
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We need to define the following concepts:
- *freshness conditions: if* x *does not occur free in* ϕ
- *substitution* $\phi[t/x]$

Observation

If logic teaches us to study reasoning,
we should also study reasoning about logics.

Logics for reasoning about logics

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Extend logics with ***explicit*** *meta-variables* $P, Q, \ldots$ *and* $T, U, \ldots$:

$$P \;\Rightarrow\; (Q \;\Rightarrow\; P)$$
$$P \;\Rightarrow\; \forall x.P \qquad \textit{if } x \textit{ does not occur free in } P$$
$$\forall x.P \;\Rightarrow\; P[T/x]$$

Side-conditions are problematic... How do we define:

- *freshness conditions "if* x *does not occur free in* P*"?*
- *substitution* $P[T/x]$*?*

Logics for reasoning about logics

Extend logics with ***explicit*** *meta-variables, freshness conditions and explicit substitutions:*

$$\vdash P \;\Rightarrow\; (Q \;\Rightarrow\; P)$$

$$\vdash_{x\#P} P \;\Rightarrow\; \forall x.P$$

$$\vdash \forall x.P \;\Rightarrow\; P[x \mapsto T]$$

Problematic side-conditions are built-in:

- $\vdash_{x\#P}$ *represents "if x does not occur free in P"*
- $P[x \mapsto T]$ *represents* $P[T/x]$

Contents of the thesis

The thesis introduces two logics to reason about logics with binders:

Equational logic:

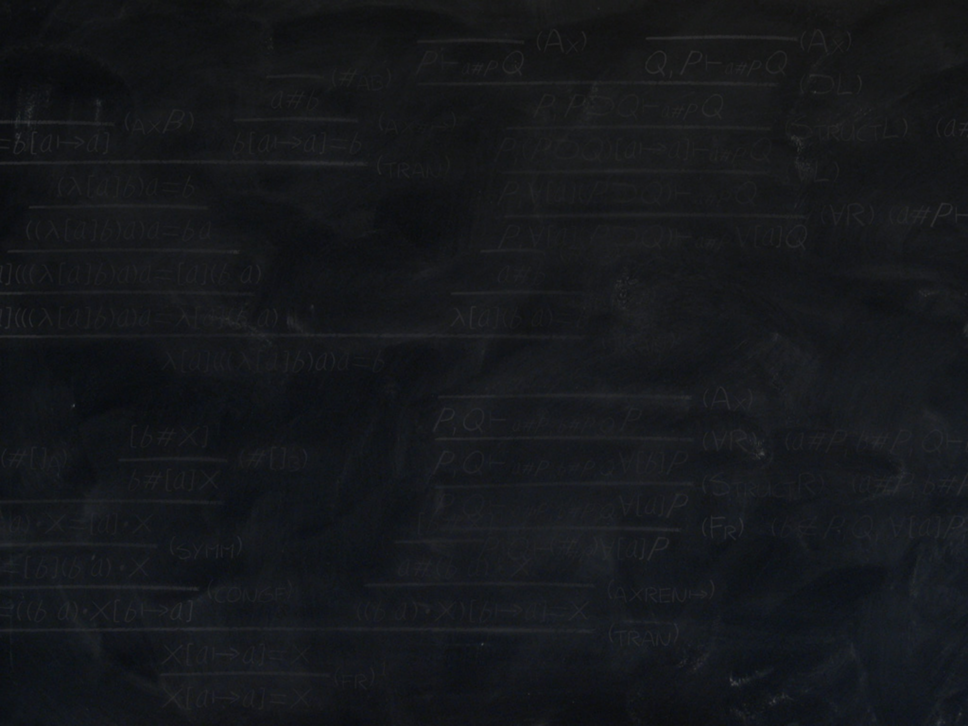
- *equational calculus (Chapter 2)*
- *semantics in nominal sets (Chapter 3)*
- *axiomatisation of explicit substitution (Chapter 4)*

First-order logic:

- *sequent calculus (Chapter 5)*
- *axiomatisation of the sequent calculus (Chapter 6)*

Conclusion

This thesis shows how we can ***formalise the meta-level*** *of* ***logics with binding*** *in a way that is* ***close to informal practice****.*



(Ax)
$P \vdash_{a\#P} Q$
(Ax)
$Q, P \vdash_{a\#P} Q$
$a\#b$
(AxB)
$b[a \mapsto a] = b$
(TRAN)
$(\lambda[a]b)a = b$
(∀R)
$\lambda[a](b\ a)$
(Ax)
(∀R)
(STRUCTR)
(FR)
$[b\#X]$
$b\#[a]X$
(SYMM)
(CONGR)
(AXREN↦)
(TRAN)
$X[a \mapsto a] = X$
(FR)
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