One-and-a-halfth-order Logic

Aad Mathijssen    Murdoch J. Gabbay

Department of Mathematics and Computer Science
Technische Universiteit Eindhoven
The Netherlands

TCS Seminar, Vrije Universiteit Amsterdam
8th September 2006
Motivation

Consider the following valid assertions in first-order logic:

- \( \phi \supset \psi \supset \phi \)
- if \( a \not\in fn(\phi) \) then \( \phi \supset \forall a.\phi \)
- if \( a \not\in fn(\phi) \) then \( \phi \supset \phi[a \mapsto t] \)
- if \( b \not\in fn(\phi) \) then \( \forall a.\phi \supset \forall b.\phi[a \mapsto b] \)

These are not valid syntax in first-order logic. This is because of meta-level concepts:

- meta-variables varying over syntax: \( \phi, \psi, a, b, t \)
- properties of syntax: \( a \not\in fn(\phi), \phi[a \mapsto t], \alpha\)-equivalence
Motivation

Consider the following valid assertions in first-order logic:

- $\phi \supset \psi \supset \phi$
- if $a \notin \text{fn}(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin \text{fn}(\phi)$ then $\phi \supset \phi[a \mapsto t]$
- if $b \notin \text{fn}(\phi)$ then $\forall a.\phi \supset \forall b.\phi[a \mapsto b]$

These are not valid syntax in first-order logic. This is because of meta-level concepts:

- meta-variables varying over syntax: $\phi$, $\psi$, $a$, $b$, $t$
- properties of syntax: $a \notin \text{fn}(\phi)$, $\phi[a \mapsto t]$, $\alpha$-equivalence
Consider the following derivations in Gentzen’s sequent calculus:

\[
\begin{align*}
\psi, \phi \vdash \phi & \quad \text{(Ax)} \quad \psi, \phi \vdash \phi \quad \text{(Ax)} \\
\phi \vdash \psi \supset \phi & \quad \text{(R)} \quad p(d), p(c) \vdash p(c) \quad \text{(R)} \\
\vdash \phi \supset \psi \supset \phi & \quad \text{(R)} \\
\vdash p(c) \supset p(d) \supset p(c) & \quad \text{(R)}
\end{align*}
\]

And for \( b \notin fn(\phi) \):

\[
\begin{align*}
\forall a. \phi \vdash \forall b. \phi[a \mapsto b] & \quad \text{(Ax)} \\
\vdash \forall a. \phi \supset \forall b. \phi[a \mapsto b] & \quad \text{(R)} \\
\end{align*}
\]
Motivation (2)

Consider the following derivations in Gentzen’s sequent calculus:

\[
\begin{align*}
&\begin{align*}
&\phi, \phi \vdash \phi \quad (\text{Ax}) \\
&\psi, \phi \vdash \phi \\
&\phi \vdash \psi \supset \phi \\
&\vdash \phi \supset \psi \supset \phi
\end{align*} \\
&\begin{align*}
&\phi \vdash \psi \supset \phi \\
&\vdash \phi \supset \psi \supset \phi
\end{align*}
\end{align*}
\]

\[
\begin{align*}
&\begin{align*}
&\phi \vdash \psi \supset \phi \\
&\phi \supset \psi \supset \phi
\end{align*} \\
&\begin{align*}
&\phi \vdash \psi \supset \phi \\
&\vdash \phi \supset \psi \supset \phi
\end{align*}
\end{align*}
\]

\[
\begin{align*}
&\begin{align*}
&p(d), p(c) \vdash p(c) \quad (\supset R) \\
&p(c) \vdash p(d) \supset p(c) \\
&\vdash p(c) \supset p(d) \supset p(c)
\end{align*} \\
&\begin{align*}
&p(d), p(c) \vdash p(c) \quad (\supset R) \\
&p(c) \vdash p(d) \supset p(c) \\
&\vdash p(c) \supset p(d) \supset p(c)
\end{align*}
\end{align*}
\]

And for \( b \not\in \text{fn}(\phi) \):

\[
\begin{align*}
&\begin{align*}
&\forall a. \phi \vdash \forall b. \phi[a \mapsto b] \quad (\text{Ax}) \\
&\forall a. \phi \vdash \forall b. \phi[a \mapsto b] \\
&\vdash \forall a. \phi \supset \forall b. \phi[a \mapsto b]
\end{align*} \\
&\begin{align*}
&\forall c. p(c) \vdash \forall d. p(d) \quad (\supset R) \\
&\forall c. p(c) \vdash \forall d. p(d) \\
&\vdash \forall c. p(c) \supset \forall d. p(d)
\end{align*}
\end{align*}
\]

The left ones are not derivations, they are schemas of derivations. The right ones might be derivations; they instances of the schemas.
Motivation (3)

Questions:

- Is there a logic in which these schematic assertions and derivations are valid syntax too?
Motivation (3)

Questions:

▶ Is there a logic in which these schematic assertions and derivations are *valid syntax* too?

▶ First-order logic and its proof systems formalise *reasoning*. But also a lot of reasoning is *about* first-order logic. So why shouldn’t that be formalised?
Motivation (3)

Questions:

▶ Is there a logic in which these schematic assertions and derivations are valid syntax too?

▶ First-order logic and its proof systems formalise reasoning. But also a lot of reasoning is about first-order logic. So why shouldn’t that be formalised?

One-and-a-halfth-order logic tries to address this by formalising:

▶ meta-variables ($\phi$, $\psi$, $a$, $b$, $t$)

▶ properties of syntax ($a \not\in fn(\phi)$, $\phi[a \mapsto t]$, $\alpha$-equivalence)
Overview

► Definition of One-and-a-halfth-order Logic
  ► Introduction
  ► Formal syntax
  ► Derivability
► Properties of One-and-a-halfth-order Logic
  ► Proof-theoretical properties
  ► Equational axiomatisation
  ► Relation to first-order logic
  ► Semantics
► Conclusions, related and future work
Introduction

In the syntax of one-and-a-halfth-order logic:

- *Unknowns* $P$, $Q$ and $T$ represent meta-variables $\phi$, $\psi$ and $t$.
- *Atoms* $a$ and $b$ represent meta-variables $a$ and $b$.
- *Freshness* $a \# P$ represents $a \notin \text{fn}(\phi)$.
- *Explicit substitution* $P[a \mapsto T]$ represents $\phi[a \mapsto t]$. 
Introduction (2)

The meta-level assertions in first-order logic

- $\phi \supset \psi \supset \phi$
- if $a \notin fn(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin fn(\phi)$ then $\phi \supset \phi[a \mapsto t]$
- if $b \notin fn(\phi)$ then $\forall a.\phi \supset \forall b.\phi[a \mapsto b]$

correspond to valid assertions in one-and-a-halfth-order logic:

- $P \supset Q \supset P$
- $a\#P \rightarrow P \supset \forall[a]P$
- $a\#P \rightarrow P \supset P[a \mapsto T]$
- $b\#P \rightarrow \forall[a]P \supset \forall[b]P[a \mapsto b]$
In sequent derivations of one-and-a-halfth-order logic:

- **Contexts of freshneses** are added to the sequents.
- **Derivability of freshneses** are added as side-conditions.
- **Substitutional equivalence on terms** is added as two derivation rules, taking care of $\alpha$-equivalence and substitution.
Introduction (4)

The (schematic) derivations in first-order logic

\[
\begin{align*}
\frac{\psi, \phi \vdash \phi}{\phi \vdash \psi \supset \phi} \quad \text{(\textit{Ax})} \\
\frac{\phi \vdash \psi \supset \phi}{\vdash \phi \supset \psi \supset \phi} \quad \text{(\textit{R})}
\end{align*}
\]

\[
\begin{align*}
\frac{p(d), p(c) \vdash p(c)}{p(c) \vdash p(d) \supset p(c)} \quad \text{(\textit{Ax})} \\
\frac{p(c) \vdash p(d) \supset p(c)}{\vdash p(c) \supset p(d) \supset p(c)} \quad \text{(\textit{R})}
\end{align*}
\]

correspond to valid derivations in one-and-a-halfth-order logic:

\[
\begin{align*}
\frac{Q, P \vdash_\emptyset P}{P \vdash_\emptyset Q \supset P} \quad \text{(\textit{Ax})} \\
\frac{P \vdash_\emptyset Q \supset P}{\vdash_\emptyset P \supset Q \supset P} \quad \text{(\textit{R})}
\end{align*}
\]

\[
\begin{align*}
\frac{p(d), p(c) \vdash_\emptyset p(c)}{p(c) \vdash_\emptyset p(d) \supset p(c)} \quad \text{(\textit{Ax})} \\
\frac{p(c) \vdash_\emptyset p(d) \supset p(c)}{\vdash_\emptyset p(c) \supset p(d) \supset p(c)} \quad \text{(\textit{R})}
\end{align*}
\]
Introduction (5)

The (schematic) derivations in first-order logic, where $b \notin fn(\phi)$,

\[
\begin{align*}
\forall a.\phi & \vdash \forall b.\phi[a \leftrightarrow b] \quad (\text{Ax}) \\
\vdash \forall a.\phi \supset \forall b.\phi[a \leftrightarrow b] & \quad (\supset \text{R})
\end{align*}
\]

correspond to valid derivations in one-and-a-halfth-order logic:

\[
\begin{align*}
\forall \llbracket a \mapsto b \rrbracket \llbracket a \mapsto c \rrbracket P & \vdash \forall a.\phi \llbracket a \mapsto c \rrbracket \llbracket a \mapsto b \rrbracket P \quad (\text{Ax}) \\
\llbracket a \mapsto c \rrbracket P & \vdash \forall \llbracket a \mapsto b \rrbracket \llbracket a \mapsto c \rrbracket P \quad (\supset \text{R})
\end{align*}
\]

(1) $b \# P \vdash_{\text{SUB}} \forall [a]P = \forall [b]P[a \leftrightarrow b]$

(2) $\emptyset \vdash_{\text{SUB}} \forall [c]p(c) = \forall [d]p(d)$
Formal syntax
Nominal terms

We use **Nominal Terms** to specify the syntax, since they have built-in support for:

- meta-variables
- binding
- freshness

Nominal terms allow for a **direct and natural** representation of systems with binding.

Nominal terms are **first-order**, not higher-order.
Formal syntax
Sorts, atoms and unknowns

**Base sorts** $F$ for ‘formulas’ and $T$ for ‘terms’.

**Atomic sort** $A$ for the object-level variables.

**Sorts** $\tau$:

$$\tau ::= F | T | A | [A]\tau$$

**Atoms** $a, b, c, \ldots$ have sort $A$.
They represent *object-level* variable symbols.

**Unknowns** $X, Y, Z, \ldots$ have sort $\tau$.
They represent *meta-level* variable symbols.
Let $P, Q, R$ be unknowns of sort $F$, and $T, U$ of sort $T$. 
We call $\pi \cdot X$ a **moderated unknown**.
This represents the **permutation of atoms** $\pi$ acting on an unknown term. Write $X$ when $\pi$ is the **identity**.

**Term-formers** are of the form $f(\tau_1, \ldots, \tau_n)_\tau$.

**Terms** $t$, subscripts indicate sorting rules:

\[
    t ::= a_A \mid (\pi \cdot X)_\tau \mid ([a_A]t_\tau)[A]_\tau \mid (f(\tau_1, \ldots, \tau_n)_\tau(t^{1}_{\tau_1}, \ldots, t^{n}_{\tau_n}))_\tau
\]

We often drop the sorting subscripts:

\[
    t ::= a \mid \pi \cdot X \mid [a]t \mid f(t_1, \ldots, t_n)
\]

Write $f$ for $f()$ if $n = 0$. 
Term-formers for one-and-a-halfth-order logic:

- $\bot(F)$: false
- $\supset(F,F)$: implication, write $\supset(\phi,\psi)$ as $\phi \supset \psi$
- $\forall([A]F)$: universal quantification, write $\forall([a]\phi)$ as $\forall[a]\phi$
- $\approx(T,T)$: object-level equality, write $\approx(t,u)$ as $t \approx u$
- $\text{var}(a)$: variable casting, write $\text{var}(a)$ as $a$
- $\text{sub}([A]\tau,T\tau)$, where $\tau \in \{T, [A]T, F, [A]F\}$: explicit substitution, write $\text{sub}([a]v, t)$ as $v[a \mapsto t]$
- $p_1(T,...,T), \ldots, p_n(T,...,T)$: object-level predicate term-formers
- $f_1(T,...,T), \ldots, f_m(T,...,T)$: object-level term-formers
Formal syntax
Terms (3)

Sugar:
\[
\begin{align*}
\top & \text{ is } \bot \cup \bot \\
\neg \phi & \text{ is } \phi \supset \bot \\
\phi \land \psi & \text{ is } \neg (\phi \supset \neg \psi) \\
\phi \lor \psi & \text{ is } \neg \phi \supset \psi \\
\phi \iff \psi & \text{ is } (\phi \supset \psi) \land (\psi \supset \phi) \\
\exists [a] \phi & \text{ is } \neg \forall [a] \neg \phi
\end{align*}
\]

Descending order of operator precedence:

\[
[a], \_ [\_ \mapsto \_], \approx, \{\neg, \forall, \exists\}, \{\land, \lor\}, \supset, \iff
\]

\land, \lor, \supset \text{ and } \iff \text{ associate to the right.}
Formal syntax
Terms (3)

Sugar:

\[ \top \text{ is } \bot \lor \bot \quad \neg \phi \text{ is } \phi \supset \bot \]
\[ \phi \land \psi \text{ is } \neg(\phi \supset \neg \psi) \quad \phi \lor \psi \text{ is } \neg \phi \supset \psi \]
\[ \phi \iff \psi \text{ is } (\phi \supset \psi) \land (\psi \supset \phi) \quad \exists[a] \phi \text{ is } \neg \forall[a] \neg \phi \]

Descending order of operator precedence:

\[ [a]_-, \_[_\leadsto _], \approx, \{\neg, \forall, \exists\}, \{\land, \lor\}, \supset, \iff \]
\[ \land, \lor, \supset \text{ and } \iff \text{ associate to the right.} \]

We may call terms of sort \( \mathbb{F} \) formulas.
Example formulas:

\[ P \supset Q \supset P \quad P \supset \forall[a]P \quad P \supset P[a \mapsto T] \quad \forall[a]P \supset \forall[b]P[a \mapsto b] \]
Formal syntax
Freshness and terms-in-context

Freshness (assertions) $a \# t$, which means ‘$a$ is fresh for $t$.
If $t$ is an unknown $X$, the freshness is called primitive.

A freshness context $\Delta$ is a set of primitive freshesses.

Example freshness contexts:

$$\emptyset \quad a \# X \quad a \# P, b \# Q$$

We call $\Delta \rightarrow t$ a term-in-context; write $t$ if $\Delta = \emptyset$. 
Terms-in-context of sort $F$ represent meta-level assertions of first-order logic. For example:

- $P \supset Q \supset P$
- $a \# P \rightarrow P \supset \forall[a]P$
- $a \# P \rightarrow P \supset P[a \mapsto T]$
- $b \# P \rightarrow \forall[a]P \supset \forall[b]P[a \mapsto b]$
Formal syntax

Assertions

Terms-in-context of sort $\mathbb{F}$ represent meta-level assertions of first-order logic. For example:

- $P \supset Q \supset P$
- $a \# P \rightarrow P \supset \forall\[a\]P$
- $a \# P \rightarrow P \supset P[a \mapsto T]$
- $b \# P \rightarrow \forall[\[a\]P \supset \forall[b]P[a \mapsto b]$

represent

- $\phi \supset \psi \supset \phi$
- if $a \notin fn(\phi)$ then $\phi \supset \forall a.\phi$
- if $a \notin fn(\phi)$ then $\phi \supset \phi[\[a \mapsto t]\]
- if $b \notin fn(\phi)$ then $\forall a.\phi \supset \forall b.\phi[\[a \mapsto b]\]$
Derivability
Sequents

Let (formula) contexts $\Phi$, $\Psi$ be finite sets of formulas. For example:

\[ \emptyset \quad \phi \quad \phi, \Phi \quad \Phi, \Phi' \]

A sequent is a triple $\Phi \vdash_{\Delta} \Psi$.
We may omit empty formula contexts, e.g. writing $\vdash_{\Delta}$ for $\emptyset \vdash_{\Delta} \emptyset$. 
Derivability
Sequent calculus

Rules resembling Gentzen’s sequent calculus for first-order logic:

\[
\begin{align*}
(Ax) & \quad \phi, \Phi \vdash \Delta \Psi, \phi \\
 \parallel & \\
(\perp L) & \quad \bot, \Phi \vdash \Delta \Psi \\
 \parallel & \\
(\top L) & \quad \phi, \Phi \vdash \Delta \Psi, \phi \top \Psi \\
 \parallel & \\
(\top R) & \quad \phi, \Phi \vdash \Delta \Psi, \psi \\
 \parallel & \\
(\forall L) & \quad \phi[a \mapsto t], \Phi \vdash \Delta \Psi \\
 \forall[a] \phi, \Phi \vdash \Delta \Psi \\
 \parallel & \\
(\forall R) & \quad \phi \vdash \Delta \Psi, \psi \\
 \Phi \vdash \Delta \Psi, \forall[a] \psi \\
 \parallel & \\
(\approx L) & \quad \phi[a \mapsto t'], \Phi \vdash \Delta \Psi \\
 t' \approx t, \phi[a \mapsto t], \Phi \vdash \Delta \Psi \\
 \parallel & \\
(\approx R) & \quad \phi \vdash \Delta \Psi, t \approx t
\end{align*}
\]
Derivability
Sequent calculus (2)

Other rules:

\[
\frac{\phi', \Phi \vdash_\Delta \Psi}{\phi, \Phi \vdash_\Delta \Psi} \quad (\text{StructL}) \\
(\Delta \vdash_{\text{SUB}} \phi' = \phi)
\]

\[
\frac{\Phi \vdash_\Delta \Psi, \psi'}{\Phi \vdash_\Delta \Psi, \psi} \quad (\text{StructR}) \\
(\Delta \vdash_{\text{SUB}} \psi' = \psi)
\]

\[
\frac{\Phi \vdash_{\Delta \cup \{a \# x_1, \ldots, a \# x_n\}} \Psi}{\Phi \vdash_\Delta \Psi} \quad (\text{Fresh}) \\
(a \notin \Phi, \Psi, \Delta)
\]

\[
\frac{\Phi \vdash_\Delta \Psi, \phi \quad \phi', \Phi \vdash_\Delta \Psi}{\Phi \vdash_\Delta \Psi} \quad (\text{Cut}) \\
(\Delta \vdash_{\text{SUB}} \phi = \phi')
\]
Derivability
Example derivations in the sequent calculus

Sequent derivation of $a\#P \rightarrow P \supset \forall[a]P$:

\[
\begin{array}{c}
\frac{P \vdash a\#P}{(Ax)} \\
\frac{P \vdash a\#P \quad (\forall R) (a\#P \vdash a\#P)}{P \vdash a\#P \quad \forall[a]P}
\end{array}
\]

\[
\frac{P \vdash a\#P \quad \forall[a]P}{(\supset R) \quad \vdash a\#P \quad P \supset \forall[a]P}
\]

Derivation of $a\#P \rightarrow P \supset P[a \mapsto T]$:

\[
\begin{array}{c}
\frac{P \vdash a\#P}{(Ax)} \\
\frac{P \vdash a\#P \quad (\text{StructR}) (a\#P \vdash_{\text{SUB}} P = P[a \mapsto T])}{P \vdash a\#P \quad P[a \mapsto T]}
\end{array}
\]

\[
\frac{P \vdash a\#P \quad P[a \mapsto T]}{(\supset R) \quad \vdash a\#P \quad P[a \mapsto T]}
\]
Derivability

Write $\Delta \vdash a \# t$ when $a \# t$ is derivable from $\Delta$ using the following inference rules:

- $\frac{a \# b}{\pi^{-1}(a) \# X}$
- $\frac{\pi^{-1}(a) \# X}{a \# \pi \cdot X}$
- $\frac{a \# t}{a \# [a] t}$
- $\frac{a \# [b] t}{\pi^{-1}(b) \# X}$
- $\frac{a \# [b] t}{a \# \pi \cdot X}$
- $\frac{a \# t \cdots a \# t_n}{a \# f(t_1, \ldots, t_n)}$

Here, $a$ and $b$ range over distinct atoms.
Derivability

Write $\Delta \vdash a \not\approx t$ when $a \not\approx t$ is derivable from $\Delta$ using the following inference rules:

$$
\begin{align*}
& \frac{a \not\approx b}{\#(ab)} & \frac{\pi^{-1}(a) \not\approx X}{a \not\approx \pi \cdot X} \\
& \frac{a \not\approx [a]t}{\#([a]a)} & \frac{a \not\approx [b]t}{\#([b]b)} & \frac{a \not\approx t_1 \cdots a \not\approx t_n}{\#(\text{f})}
\end{align*}
$$

Here, $a$ and $b$ range over distinct atoms.

Examples:

$$
\begin{align*}
\vdash a \not\approx b & \quad \vdash a \not\approx \forall[a]P & \quad a \not\approx P \vdash a \not\approx \forall[b]P
\end{align*}
$$
Equality (assertions) \( t = u \), where \( t \) and \( u \) are of the same sort. Write \( \Delta \vdash_{\text{SUB}} t = u \) when \( t = u \) is derivable from \( \Delta \) using the following inference rules, where \( A \) are axioms from SUB only:

\[
\begin{align*}
&\text{(refl)} \quad \frac{t = t}{t = t} \\
&\text{(symm)} \quad \frac{t = u}{u = t} \\
&\text{(tran)} \quad \frac{t = u}{u = v} \\
&\text{(cong)} \quad \frac{t = u}{C[t] = C[u]} \\
&\text{(perm)} \quad \frac{a\#t \quad b\#t}{(a \ b) \cdot t = t} \\
&\text{(axA)} \quad \frac{\Delta^\pi \sigma}{t^\pi \sigma = u^\pi \sigma} \quad A \text{ is } \Delta \rightarrow t = u
\end{align*}
\]
Axioms of theory SUB:

- \((\text{var} \mapsto)\) \(a[a \mapsto T] = T\)
- \((\# \mapsto)\) \(a\#X \rightarrow X[a \mapsto T] = X\)
- \((f \mapsto)\) \(f(X_1, \ldots, X_n)[a \mapsto T] = f(X_1[a \mapsto T], \ldots, X_n[a \mapsto T])\)
- \((\text{abs} \mapsto)\) \(b\#T \rightarrow ([b]X)[a \mapsto T] = [b](X[a \mapsto T])\)
- \((\text{ren} \mapsto)\) \(b\#X \rightarrow X[a \mapsto b] = (b \ a) \cdot X\)
Derivability

Equality (2)

Axioms of theory SUB:

- ((var \mapsto) a[a \mapsto T] = T)
- ((\# \mapsto) a\#X \rightarrow X[a \mapsto T] = X)
- ((f \mapsto) f(X_1, \ldots, X_n)[a \mapsto T] = f(X_1[a \mapsto T], \ldots, X_n[a \mapsto T]))
- ((abs \mapsto) b\#T \rightarrow ([b]X)[a \mapsto T] = [b](X[a \mapsto T]))
- ((ren \mapsto) b\#X \rightarrow X[a \mapsto b] = (b \ a) \cdot X)

Examples:

$b\#P \vdash_{SUB} \forall[a]P = \forall[b]P[a \mapsto b]$

$\vdash_{SUB} X[a \mapsto a] = X$

$a\#Y \vdash_{SUB} Z[a \mapsto X][b \mapsto Y] = Z[b \mapsto Y][a \mapsto X[b \mapsto Y]]$
Derivability
Equality (2)

Axioms of theory SUB:

- \((\text{var} \mapsto)\) \quad \quad a[a \mapsto T] = T
- \((\# \mapsto)\) \quad \quad a\#X \rightarrow X[a \mapsto T] = X
- \((f \mapsto)\) \quad \quad f(X_1, \ldots, X_n)[a \mapsto T] = f(X_1[a \mapsto T], \ldots, X_n[a \mapsto T])
- \((\text{abs} \mapsto)\) \quad \quad b\#T \rightarrow ([b]X)[a \mapsto T] = [b](X[a \mapsto T])
- \((\text{ren} \mapsto)\) \quad \quad b\#X \rightarrow X[a \mapsto b] = (b \ a) \cdot X

Examples:

\[
\begin{align*}
  b\#P \vdash_{\text{SUB}} \forall [a]P &= \forall [b]P[a \mapsto b] \\
  \vdash_{\text{SUB}} X[a \mapsto a] &= X \\
  a\#Y \vdash_{\text{SUB}} Z[a \mapsto X][b \mapsto Y] &= Z[b \mapsto Y][a \mapsto X[b \mapsto Y]]
\end{align*}
\]

Nominal Algebra is the theory of equality on nominal terms.
Proof-theoretical properties
Permutation and instantiation

We may permute atoms and instantiate unknowns in derivations.

**Theorem**

*If \( \Pi \) is a valid derivation of \( \Phi \vdash_\Delta \Psi \), then \( \Pi^\pi \) is a valid derivation of \( \Phi^\pi \vdash_{\Delta^\pi} \Psi^\pi \).*

**Theorem**

*If \( \Pi \) is a valid derivation of \( \Phi \vdash_\Delta \Psi \) and \( \Delta' \vdash \Delta_\sigma \), then \( \Pi(\sigma, \Delta') \) is a valid derivation of \( \Phi_\sigma \vdash_{\Delta'} \Psi_\sigma \).*

\( \Pi(\sigma, \Delta') \) is \( \Pi \) in which:

- each unknown \( X \) is replaced by \( \sigma(X) \)
- each freshness context \( \Delta \) is replaced by \( \Delta' \)
Proof-theoretical properties
Instantiation example

Take the following derivations:

\[
\begin{align*}
(P \vdash a \# P) & \quad (Ax) \\
(P \vdash a \# P) & \quad (StructR) \ (1) \\
(P \vdash a \# P) & \quad (\supset R) \\
\vdash a \# P & \quad (\supset R)
\end{align*}
\]

\[
\begin{align*}
(p(c) \vdash \emptyset) & \quad (Ax) \\
(p(c) \vdash \emptyset) & \quad (StructR) \ (2) \\
(p(c) \vdash \emptyset) & \quad (\supset R) \\
\vdash \emptyset & \quad (\supset R)
\end{align*}
\]

(1) \( a \# P \vdash_{\text{SUB}} P = P[a \mapsto T] \)
(2) \( \emptyset \vdash_{\text{SUB}} p(c) = p(c)[a \mapsto d] \)

The derivation on the right is an instance of the one on the left:

- call the left derivation \( \Pi \)
- then the right one is \( \Pi([p(c)/P, d/T], \emptyset) \), which is valid because \( \emptyset \vdash a \# P[p(c)/P, d/T] \), i.e. \( \emptyset \vdash a \# p(c) \)
Proof-theoretical properties

Cut elimination

Theorem (Cut elimination)

*The (Cut) rule is admissible in the system without it.*
Proof-theoretical properties

Cut elimination

Theorem (Cut elimination)

*The (Cut) rule is admissible in the system without it.*

Corollary

*The sequent calculus is consistent, i.e. $\vdash_\Delta$ can never be derived.*
Axiomatisation
Theory FOL

Theory FOL extends theory SUB with the following axioms:

\[
P \supset Q \supset P = \top \quad \neg P \supset P = \top \quad \top \supset P = P \quad \text{(Props)}
\]

\[
(P \supset Q) \supset (Q \supset R) \supset (P \supset R) = \top \quad \bot \supset P = \top
\]

\[
\forall[a] P \supset P[a \mapsto T] = \top \quad \text{(Quants)}
\]

\[
\forall[a](P \land Q) \iff \forall[a]P \land \forall[a]Q = \top
\]

\[
a \# P \rightarrow \forall[a](P \supset Q) \iff P \supset \forall[a]Q = \top
\]

\[
T \approx T = \top \quad U \approx T \land P[a \mapsto T] \supset P[a \mapsto U] = \top \quad \text{(Eq)}
\]

Axioms of the form \( \phi = \top \) intuitively mean ‘\( \phi \) is true’. Note that this is a finite number of axioms.
Axiomatisation
Equivalence with sequent calculus

Sequent and equational derivability are equivalent:

**Theorem**

*For all formula contexts \( \Phi, \Psi \) and freshness contexts \( \Delta \):*

\[
\Phi \vdash_\Delta \Psi \text{ is derivable} \iff \Delta \vdash_{\text{FOL}} \Phi^\wedge \supset \Psi^\vee = \top.
\]

Here:

- \( \Phi^\wedge \) is the *conjunction* of all formulas in \( \Phi \)
- \( \Psi^\vee \) the *disjunction* of all formulas in \( \Psi \)
Axiomatisation
Equivalence with sequent calculus

Sequent and equational derivability are equivalent:

**Theorem**
*For all formula contexts* $\Phi, \Psi$ *and freshness contexts* $\Delta$:

\[ \Phi \vdash_{\Delta} \Psi \text{ is derivable} \iff \Delta \vdash_{\text{FOL}} \Phi^\land \supset \Psi^\lor = \top. \]

Here:

- $\Phi^\land$ is the *conjunction* of all formulas in $\Phi$
- $\Psi^\lor$ the *disjunction* of all formulas in $\Psi$

**Corollary**
*Theory* FOL *is consistent, i.e.* $\Delta \vdash_{\text{FOL}} \top = \bot$ *does not hold.*
Relation to First-order Logic

Call a term or a formula context ground if it does not contain unknowns or explicit substitutions.

Call $\Phi \vdash \Psi$ a first-order sequent when $\Phi$ and $\Psi$ are ground. Gentzen’s sequent calculus for first-order logic:

\[
\begin{align*}
\text{(Ax)} & \quad \phi, \Phi \vdash \psi, \phi \\
\text{(⊥L)} & \quad \bot, \Phi \vdash \psi \\
\text{(⊃L)} & \quad \phi, \Phi \vdash \psi, \phi \supset \psi \\
\text{(∀L)} & \quad \phi[[a \mapsto t]], \Phi \vdash \psi, \forall a. \phi \\
\text{(∀R)} & \quad \phi \vdash \psi, \phi \supset \psi, \forall a. \phi \\
\text{(≈L)} & \quad \phi[[a \mapsto t'], \Phi \vdash \psi, t \approx t, \phi[[a \mapsto t]], \Phi \vdash \psi \\
\text{(≈R)} & \quad \phi \vdash \psi, t \approx t
\end{align*}
\]
Relation to First-order Logic (2)

Note that:

- we write $\forall a.\phi$ for $\forall[a]\phi$
- $[a \mapsto t]$ is capture-avoiding substitution
- $a \not\in fn(\phi)$ is ‘$a$ does not occur in the free names of $\phi$’
- we take formulas up to $\alpha$-equivalence
Relation to First-order Logic (2)

Note that:
- we write $\forall a. \phi$ for $\forall [a] \phi$
- $[a \mapsto t]$ is capture-avoiding substitution
- $a \not\in fn(\phi)$ is ‘$a$ does not occur in the free names of $\phi$’
- we take formulas up to $\alpha$-equivalence

On ground terms, one-and-a-halfth-order logic is first-order logic:

**Theorem**

$\Phi \vdash \Psi$ is derivable in the sequent calculus for first-order logic, iff

$\Phi \vdash_\emptyset \Psi$ is derivable in the sequent calculus for one-and-a-halfth-order logic.
Semantics

For closed terms $t$, its ground form $t[]$ is $t$ in which each explicit substitution $v[a \mapsto u]$ is replaced by $v[a \mapsto u]$.

Lemma

For closed terms $t$, $\vdash_{\text{SUB}} t = t[]$.

A term-in-context $\Delta \rightarrow \phi$ is valid iff $\phi\sigma[]$ is valid in first-order logic for all instantiations $\sigma$ such that $\phi\sigma$ is closed and $\vdash \Delta\sigma$ holds.
Semantics

For closed terms $t$, its ground form $t[]$ is $t$ in which each explicit substitution $v[a \mapsto u]$ is replaced by $v[a \mapsto u]$.

Lemma

For closed terms $t$, $\vdash \text{SUB} t = t[]$.

A term-in-context $\Delta \rightarrow \phi$ is valid iff $\phi\sigma[]$ is valid in first-order logic for all instantiations $\sigma$ such that $\phi\sigma$ is closed and $\vdash \Delta\sigma$ holds.

The sequent calculus for one-and-a-halfth-order logic is sound for this semantics:

Theorem

If $\vdash_{\Delta} \phi$ is derivable then $\Delta \rightarrow \phi$ is valid.
Conclusions

Using nominal terms, we can:

▶ accurately represent systems with binding:
  e.g. explicit substitution and first-order logic
▶ specify novel systems with their own mathematical interest:
  e.g. one-and-a-halfth-order logic

One-and-a-halfth-order logic:

▶ makes meta-level concepts of first-order logic explicit
▶ has a sequent calculus with syntax-directed rules
▶ has a semantics in first-order logic
▶ has a finite equational axiomatisation
▶ is the result of axiomatising first-order logic in nominal algebra
Related work

In **Second-Order logic (SOL)** we can quantify over predicates anywhere: more expressive than one-and-a-half-order logic.

On the other hand, we can easily extend theory FOL with *one* axiom to express the principle of induction on natural numbers:

\[
P[a \mapsto 0] \land \forall[a](P \supset P[a \mapsto succ(a)]) \supset \forall[a]P = \top.
\]

**Higher-Order Logic (HOL)** is type raising, while our logic is *not*:

- \( P[a \mapsto t] \) corresponds to \( f(t) \) in HOL, where \( f : \mathbb{T} \to \mathbb{F} \)
- \( P[a \mapsto t][a' \mapsto t'] \) corresponds to \( f'(t)(t') \) where \( f' : \mathbb{T} \to \mathbb{T} \to \mathbb{F} \)

One-and-a-halfth-order logic is not a subset of SOL or HOL because of freshesses.
Future work

Topics:

- Completeness of the sequent calculus with respect to the semantics.
- Let unknowns range over sequent derivations, and establish a Curry-Howard correspondence (term-in-contexts as types, derivations as terms).
- Two-and-a-halfth-order logic (where you can abstract $X$)?
- Implementation and automation?
Further reading

Murdoch J. Gabbay, Aad Mathijssen: One-and-a-halfth-order Logic. PPDP’06.

Murdoch J. Gabbay, Aad Mathijssen: Capture-Avoiding Substitution as a Nominal Algebra. ICTAC’06.

Murdoch J. Gabbay, Aad Mathijssen: Nominal Algebra. Submitted STACS’07.
Just to scare you

\[
P[b \leftrightarrow c][a \leftrightarrow c] \vdash_{c \# P} P[b \leftrightarrow c][a \leftrightarrow c]
\]

(Ax)

\[
\forall [a]P[b \leftrightarrow c] \vdash_{c \# P} P[b \leftrightarrow c][a \leftrightarrow c]
\]

(\forall L)

\[
(\forall [a]P)[b \leftrightarrow c] \vdash_{c \# P} P[b \leftrightarrow a][a \leftrightarrow c]
\]

(StructL) (1)

\[
\forall [b]\forall [a]P \vdash_{c \# P} P[b \leftrightarrow c][a \leftrightarrow c]
\]

(\forall L)

\[
\forall [b]\forall [a]P \vdash_{c \# P} \forall [c]P[b \leftrightarrow c][a \leftrightarrow c]
\]

(\forall R) (2)

\[
\forall [b]\forall [a]P \vdash_{c \# P} \forall [a]P[b \leftrightarrow a]
\]

(StructR) (3)

\[
\forall [b]\forall [a]P \vdash_{\emptyset} \forall [a]P[b \leftrightarrow a]
\]

(Fresh) (4)

Side-conditions:
(1) \(c \# P \vdash_{\text{SUB}} \forall [a]P[b \leftrightarrow c] = (\forall [a]P)[b \leftrightarrow c]\)
(2) \(c \# P \vdash c \# \forall [b]\forall [a]P\)
(3) \(c \# P \vdash_{\text{SUB}} \forall [c]P[b \leftrightarrow c][a \leftrightarrow c] = \forall [a]P[b \leftrightarrow a]\)
(4) \(c \not\in \forall [b]\forall [a]P, \forall [a]P[b \leftrightarrow a]\)